MA922 SOLVING TRIGONOMETRIC EQUATIONS

(1) Find all solutions of the equation

$$2\sin x - \sqrt{3} = 0$$

Solution: First we solve for $\sin x$:

$$\sin x = \frac{\sqrt{3}}{2}$$

Next we find all solutions of this equation in the interval $[0, 360°]$.

$$x = \arcsin(\frac{\sqrt{3}}{2}) = 60°$$

So the first solution is $60°$. The other solution corresponds to a point in II quadrant (since such a point has positive y coordinate). It has a reference angle of $60°$. Thus, it is equal to $180° - 60° = 120°$.

Finally we find all solutions. All angles that correspond to the point determined by $60°$ angle are

$$60° + k \cdot 360°$$

where $k = 0, \pm 1, \pm 2, \ldots$. All angles that correspond to the point determined by 120^o angle are

$$120° + k \cdot 360°$$

where $k = 0, \pm 1, \pm 2, \ldots$. We note that k gives the number of full rotations around the circle. If k is positive, the rotations are in counterclockwise direction, if k is negative, the rotations are in clockwise direction

(2) Find all solutions of the equation in $[0, 360°]$.

$$\sqrt{3}\sin 2x = \cos 2x$$

Solution: The objective is to get an equation of the form: trigonometric function = number. To this end, it is convenient to divide by $\cos 2x$:

$$\sqrt{3}\tan 2x = 1.$$

Next we divide by $\sqrt{3}$:

$$\tan 2x = \frac{1}{\sqrt{3}}$$

Let $\theta = 2x$. We get a standard equation

$$\tan\theta = \frac{1}{\sqrt{3}}.$$

We find ALL solutions of this equation. As a first step we find all solutions for θ in the interval $[0, 360]^{\circ}$.

$$\theta_1 = 30°$$

$$\theta_2 = 210°$$

All angles that correspond to the point determined by $30°$ angle are

$$\theta_1 = 30° + k \cdot 360°$$

where $k = 0, \pm1, \pm2, \ldots$.All angles that correspond to the point determined by $210°$ angle are

$$\theta_2 = 210° + k \cdot 360°$$

where $k = 0, \pm1, \pm2, \ldots$. To find all solutions for the angle x we recall that $\theta = 2x$. Thus,

$$2x_1 = 30° + k \cdot 360°$$

Dividing by 2:

$$x_1 = 15° + k \cdot 180°$$

From here for $k = 0$ we get $15°$ which is in $[0, 360°]$. For $k = 1$ we get $195°$ which is also in $[0, 360°]$. For the other values of k we get solutions outside the interval $[0, 360°]$. The second set of solutions is gives

$$2x_2 = 210° + k \cdot 360°$$

Dividing by 2:

$$x_2 = 105° + k \cdot 180°$$

From here for $k = 0$ we get $105°$ which is in $[0, 360°]$. For $k = 1$ we get $285°$ which is also in $[0, 360°]$. For the other values of k we get solutions outside the interval $[0, 360°]$. Thus, all solutions in the interval $[0, 360°]$ are $15°, 105°, 195°, 285°$.

(3) Find all solutions of the equation in $[0, 360°]$.

$$2 \sin 2x = \tan 2x$$

Solution: This equation is very similar to the previous one, however one should be careful here. It is not wise to divide by $\sin 2x$ or by $\tan 2x$. In general when solving trigonometric equations: DO NOT CANCEL (or divide). This leads to loss of solutions. We could do that in the previous example because no solutions were lost this way. However, if we do that here we will lose solutions. So instead, move everything to one side and FACTOR. We have

$$2 \sin 2x - \frac{\sin 2x}{\cos 2x} = 0$$

Factoring out $\sin 2x$ we get

$$\sin 2x \left(2 - \frac{1}{\cos 2x}\right) = 0$$

Solving this equation reduces to solving the following 2 equations

$$\sin 2x = 0 \qquad \text{and} \qquad \frac{1}{\cos 2x} = 2$$

Consider the first of these two equations

$$\sin 2x = 0.$$

To solve this equation we set $\theta = 2x$ and we find all solutions of $\sin\theta = 0$. All solutions of this equation are

$$\theta = k \cdot 180^\circ$$

where $k = 0, \pm 1, \pm 2, \ldots$. (Consult the graph of the function $y = \sin\theta$.) Thus,

$$2x = k \cdot 180^\circ$$

So we have

$$x = k \cdot 90^\circ$$

We give values to k to see which of these solutions belong to $[0, 360^\circ]$. If $k = 0$ we get 0° which belongs to $[0, 360^\circ]$. If $k = 1$ we get 90^o. If $k = 2$ we get 180°. If $k = 3$ we get 270°. If $k = 4$ we get 360°. (If we have cancelled $\sin 2x$ we would have lost these solutions.)

The other equation is

$$\frac{1}{\cos 2x} = 2$$

or, equivalently,

$$\cos 2x = \frac{1}{2}.$$

We set $\theta = 2x$ and we have to find all solution to the equation

$$\cos\theta = \frac{1}{2}$$

There are 2 solutions of this equation in $[0, 360^\circ]$ which give 2 distinct points on the unit circle. $\theta_1 = 60^\circ$ and $\theta_2 = 300^\circ$. All angles that correspond to the fist point are

$$\theta_1 = 60^\circ + k \cdot 360^\circ$$

From here we find the values of x:

$$2x_1 = 60^\circ + k \cdot 360^\circ$$

Dividing

$$x_1 = 30^\circ + k \cdot 180^\circ$$

We give values of k to see which of these are in $[0, 360^\circ]$: For $k = 0$ we get 30° which is a solution. For $k = 1$ we get 210° which is another solution. All angles that correspond to the second point are

$$\theta_2 = 300^\circ + k \cdot 360^\circ$$

Thus,

$$x_1 = 150^\circ + k \cdot 180^\circ$$

For $k = 0$ we get 150°. For $k = 1$ we get 330°. So we obtain the following solutions to this problem: 0°, 90°, 180°, 270°, 360°, 30°,210°,150°, 330°.

(4) Find all solutions of the equation in $[0, 2\pi]$.

$$\sin(-x) + 3\sin x = 1$$

Solution: To solve a trigonometric equation, normally we should have only one angle present. Thus we use the fact that $\sin(-x) = -\sin x$ to get

$$-\sin x + 3\sin x = 1$$

or equivalently,

$$2\sin x = 1$$

We get the standard form

$$\sin x = \frac{1}{2}$$

The solutions of this equation in $[0, 2\pi]$ are $\frac{\pi}{6}$ and $\frac{5\pi}{6}$.

(5) Find all solutions of the equation in $[0, 360°]$.

$$\sin^2 x = 4 - 2\cos^2 x \tag{1}$$

Solution: We want only one of the functions $\sin x$ and $\cos x$ to be present. To get this we use the identity

$$\cos^2 x + \sin^2 x = 1$$

From here we get

$$\sin^2 x = 1 - \cos^2 x$$

Using this identity in equation (1) we get

$$1 - \cos^2 x = 4 - 2\cos^2 x$$

Solving for $\cos^2 x$ we get

$$\cos^2 x = 3$$

Taking a square root we get 2 equations from this one. The first one is

$$\cos x = \sqrt{3}$$

which has no solutions and

$$\cos x = -\sqrt{3}$$

which also has no solutions.

(6) Find all exact solutions of the equation in $[-2b, 2b]$. Here b is a positive constant.

$$2\sin\left(\frac{\pi\theta}{b}\right) = \sqrt{2} \tag{2}$$

Solution: First we solve for $\sin\left(\frac{\pi\theta}{b}\right)$

$$\sin\left(\frac{\pi\theta}{b}\right) = \frac{\sqrt{2}}{2}$$

Let $\alpha = \frac{\pi\theta}{b}$. Thus we first have to find ALL solutions of

$$\sin\alpha = \frac{\sqrt{2}}{2}.$$

To find all solutions of this equation we first have to find to find all solutions in $[0, 2\pi]$. The solutions for α are $\alpha_1 = \frac{\pi}{4}$ and $\alpha_2 = \frac{3\pi}{4}$. Each of these angles determines one point on the unit circle. All angles that correspond to the first point are

$$\alpha_1 = \frac{\pi}{4} + k \cdot 2\pi$$

where $k = 0, \pm 1, \pm 2, \ldots$. Next we have to find the solutions in terms of the angle θ:

$$\frac{\pi\theta_1}{b} = \frac{\pi}{4} + k \cdot 2\pi$$

Solving for θ_1:

$$\theta_1 = \frac{b}{\pi} \cdot \frac{\pi}{4} + \frac{b}{\pi} \cdot k \cdot 2\pi$$

Simplifying,

$$\theta_1 = \frac{b}{4} + 2bk$$

To get specific solutions we give values of to k. For $k = 0$ we get $\frac{b}{4}$ which is in the interval $[-2b, 2b]$. Next we let $k = 1$ we get $\frac{9b}{4}$ which is outside the interval $[-2b, 2b]$. If we give larger values of b we will only get larger solutions which are outside the specified interval. Next we take $k = -1$ and we get $\frac{-7b}{4}$ which is in the interval $[-2b, 2b]$ and is a solution. If we give smaller values of k we will go outside the interval.

Next we repeat the process with all angles α corresponding to the second point:

$$\alpha_2 = \frac{3\pi}{4} + k \cdot 2\pi$$

where $k = 0, \pm 1, \pm 2, \ldots$. Next we have to find the solutions in terms of the angle θ:

$$\frac{\pi\theta_2}{b} = \frac{3\pi}{4} + k \cdot 2\pi$$

Solving for θ_2:

$$\theta_2 = \frac{b}{\pi} \cdot \frac{3\pi}{4} + \frac{b}{\pi} \cdot k \cdot 2\pi$$

Simplifying,

$$\theta_2 = \frac{3b}{4} + 2bk$$

To get specific solutions we give values of to k. For $k = 0$ we get $\frac{3b}{4}$ which is in the interval $[-2b, 2b]$. Next we let $k = 1$ we get $\frac{11b}{4}$ which is outside the interval $[-2b, 2b]$. If we give larger values of k we will only get larger solutions which are outside the specified interval. Next we take $k = -1$ and we get $\frac{-5b}{4}$ which is in the interval $[-2b, 2b]$ and is a solution. If we give smaller values of k we will go outside the interval. So all solutions to the problem are

$$\frac{b}{4}, \frac{-7b}{4}, \frac{3b}{4}, \frac{-5b}{4}.$$

(7) Find all exact solutions of the equation in $[0, 2\pi]$.

(3) $$\sin^2 x + \frac{1}{2} = \sqrt{2}\sin x$$

Solution: When we have powers and/or products of trigonometric functions we have to factor. Set $y = \sin x$. Then the equation becomes

$$y^2 - \sqrt{2}y + \left(\frac{1}{\sqrt{2}}\right)^2 = 0$$

Realizing that this is equivalent to

$$\left(y - \frac{1}{\sqrt{2}}\right)^2 = 0$$

Going back to to $\sin x$ we have

$$\left(\sin x - \frac{1}{\sqrt{2}}\right)^2 = 0$$

From here we get a unique equation

$$\sin x = \frac{1}{\sqrt{2}}$$

The solutions of this equation in $[0, 2\pi]$ are $\frac{\pi}{4}$ and $\frac{3\pi}{4}$.

(8) Find all exact solutions of the equation in $[0, 2\pi]$.

(4) $$2\sin^2 x - \cos x = 1$$

Solution: This problem is similar to the one before except for the fact that the trigonometric function which is squared is $\sin x$ while the one which is not raised to a power is $\cos x$. In order to view this as a quadratic equation we need to have the same function in both cases. The function that must be chosen should be $\cos x$ since it is easier to express $\sin^2 x$ in terms of $\cos^2 x$ that to express $\cos x$ in terms of $\sin x$. Therefore, by the Pythagorean identity we have

$$\sin^2 x = 1 - \cos^2 x.$$

We use that in (4):

$$2(1 - \cos^2 x) - \cos x = 1$$

Simplifying and writing this as a quadratic equation we get

$$2\cos^2 x + \cos x - 1 = 0$$

Factoring this expression as a quadratic equation ($2y^2 + y - 1 = 0$):

$$(2\cos x - 1)(\cos x + 1) = 0$$

Thus we obtain 2 equations that we have to solve separately:

$$2\cos x - 1 = 0 \qquad \text{and} \cos x = -1$$

Considering the first of these equations we have $\cos x = \frac{1}{2}$ whose solutions are $\dfrac{\pi}{3}$ and $\dfrac{5\pi}{3}$. The second equation is

$$\cos x = -1$$

whose solution in $[0, 2\pi]$ is π. So the problem has 3 solutions: $\dfrac{\pi}{3}$, $\dfrac{5\pi}{3}$ and π.

(9) Find all exact solutions of the equation in $[0, 2\pi]$.

$$\sin 2\pi x + \sin \pi x = 0 \tag{5}$$

Solution: Since both the angles $2\pi x$ and πx are involved, and we would like to have only one angle involved, it is useful to use the double-angle formula for the sine. In particular, we have,

$$\sin 2\pi x = 2 \sin \pi x \cos \pi x.$$

Thus from equation (5) we have

$$2 \sin \pi x \cos \pi x + \sin \pi x = 0$$

We factor out $\sin \pi x$ to obtain

$$\sin \pi x (2 \cos \pi x + 1) = 0$$

This equation reduces to 2 simpler equations

$$\sin \pi x = 0 \qquad \text{and} \qquad 2 \cos \pi x + 1 = 0.$$

Consider first the equation $\sin \pi x = 0$. Set $\theta = \pi x$. All solution to the equation

$$\sin \theta = 0$$

are

$$\theta = k \cdot \pi$$

where $k = 0, \pm 1, \pm 2, \ldots$. Hence

$$\pi x = k \cdot \pi$$

and consequently

$$x = k$$

where $k = 0, \pm 1, \pm 2, \ldots$. The solutions in the interval $[0, 2\pi]$ are $x = 0, 1, 2, 3, 4, 5, 6$.

The second equation to consider is

$$2 \cos \pi x + 1 = 0$$

Solving for the cosine we obtain

$$\cos \pi x = -\frac{1}{2}$$

Set $\theta = \pi x$. The equation above becomes

$$\cos \theta = -\frac{1}{2}$$

All solutions of this equation are

$$\theta_1 = \frac{2\pi}{3} + k \cdot 2\pi$$

where $k = 0, \pm 1, \pm 2, \dots$. Recall that $\theta = \pi x$. Hence

$$\pi x_1 = \frac{2\pi}{3} + k \cdot 2\pi$$

$$x_1 = \frac{2}{3} + 2k$$

Giving values to k we obtain the solutions which are in the interval $[0, 2\pi]$: $x = \frac{2}{3}, \frac{8}{3}, \frac{14}{3}$. The other set of solutions of the equation

$$\cos\theta = -\frac{1}{2}$$

is

$$\theta_2 = \frac{4\pi}{3} + k \cdot 2\pi$$

where $k = 0, \pm 1, \pm 2, \dots$. Hence,

$$x_2 = \frac{4}{3} + 2k$$

Giving values to k we obtain the solutions in the interval $[0, 2\pi]$ $x = \frac{4}{3}, \frac{10}{3}, \frac{16}{3}$.

So all answers to the problem are: $x = 0, 1, 2, 3, 4, 5, 6, \frac{2}{3}, \frac{8}{3}, \frac{14}{3}, \frac{4}{3}, \frac{10}{3}, \frac{16}{3}$.

(10) Find all exact solutions of the equation in $[0, 2\pi]$.

(6) $$\cos 2x + \cos x = 2$$

Solution: As in the previous problem we would like to have only the angle x involved so we use a double-angle formula for the cosine. Since the other function is also cosine we should use the double-angle formula that relates the cosine to cosine only:

$$\cos 2x = 2\cos^2 x - 1$$

Thus from (6) we have

$$2\cos^2 x - 1 + \cos x = 2$$

Rewriting this as a quadratic equation:

$$2\cos^2 x + \cos x - 3 = 0$$

Factoring we have

$$(2\cos x + 3)(\cos x - 1) = 0$$

This equation reduces to the following 2 equations:

$$2\cos x + 3 = 0 \qquad \text{and} \qquad \cos x = 1.$$

The first one has no solutions since $\cos x$ cannot be equal to $-\frac{3}{2}$.

The second equation is

$$\cos x = 1$$

whose solutions in $[0, 2\pi]$ are $x = 0$ and $x = 2\pi$.

(11) Find all exact solutions of the equation in $[0, 2\pi]$.

(7)
$$\cos 4x + \cos x = 0$$

Solution: This equation is somewhat similar to the previous one. However here it is not convenient to apply double angle formula to $\cos 4x$ twice because we will obtain an equation which involves powers of $\cos x$ higher than 2. Here it is more convenient to apply an appropriate formula from those in the box on page 309. In particular, the formula of cosine plus cosine is the one we need. We have

$$\cos 4x + \cos x = 2\cos\frac{5x}{2}\cos\frac{3x}{2}$$

Thus the equation becomes

$$2\cos\frac{5x}{2}\cos\frac{3x}{2} = 0$$

which reduces to solving 2 equations

$$\cos\frac{5x}{2} = 0 \qquad\qquad \cos\frac{3x}{2} = 0$$

Since all solutions of the equation

$$\cos\theta = 0$$

are

$$\theta = \frac{\pi}{2} + k\cdot\pi$$

we have

$$\frac{5x}{2} = \frac{\pi}{2} + k\cdot\pi$$

Solving for x we get

$$x = \frac{\pi}{5} + k\cdot\frac{2\pi}{5}$$

Giving values to k we obtain solutions in the interval $[0, 2\pi]$. For $k = 0$ we obtain $\frac{\pi}{5}$, for $k = 1$, we obtain $\frac{3\pi}{5}$, for $k = 2$, we obtain π, for $k = 3$, we obtain $\frac{7\pi}{5}$, for $k = 4$, we obtain $\frac{9\pi}{5}$. For the other values of k we obtain solutions which are not in the interval $[0, 2\pi]$. For the second equation

$$\cos\frac{3x}{2} = 0$$

we obtain

$$\frac{3x}{2} = \frac{\pi}{2} + k\cdot\pi$$

Solving for x we get

$$x = \frac{\pi}{3} + k\cdot\frac{2\pi}{3}$$

Giving values to k we obtain solutions in the interval $[0, 2\pi]$. For $k = 0$ we obtain $\frac{\pi}{3}$, for $k = 1$, we obtain π, for $k = 2$, we obtain $\frac{5\pi}{3}$, For the other values of k we obtain solutions which are not in the interval $[0, 2\pi]$. Thus, all solutions of the problem are $\frac{\pi}{5}$, $\frac{3\pi}{5}$,π, $\frac{7\pi}{5}$, $\frac{9\pi}{5}$, $\frac{\pi}{3}$, $\frac{5\pi}{3}$.

(12) Find all solutions of the equation in $[-1, 1]$.

(8) $$\sin(2\pi x) + \sqrt{3}\cos(2\pi x) = 0$$

Solution: Because the right-hand side is zero, we can move $\sqrt{3}\cos(2\pi x)$ to the right-hand side and divide by $\cos(2\pi x)$:

$$\sin(2\pi x) = -\sqrt{3}\cos(2\pi x)$$

Dividing by the cosine

$$\frac{\sin(2\pi x)}{\cos(2\pi x)} = -\sqrt{3}$$

Thus we obtain the equation

$$\tan(2\pi x) = -\sqrt{3}$$

Let $\theta = 2\pi x$. Then we solve the equation

$$\tan(\theta) = -\sqrt{3}$$

whose solutions are

$$\theta_1 = \frac{2\pi}{3} + k \cdot 2\pi \qquad \theta_2 = \frac{5\pi}{3} + k \cdot 2\pi$$

Going back to x

$$2\pi x_1 = \frac{2\pi}{3} + k \cdot 2\pi$$

Dividing by 2π:

$$x_1 = \frac{1}{3} + k$$

For $k = -1$ we obtain $-\frac{2}{3}$. For $k = 0$ we get $\frac{1}{3}$. For the other values of k we don't obtain solutions in the interval $[-1, 1]$. From θ_2 we have

$$2\pi x_1 = \frac{5\pi}{3} + k \cdot 2\pi$$

Dividing by 2π

$$x_2 = \frac{5}{6} + k$$

Thus, for $k = -1$ we get $-\frac{1}{6}$. For $k = 0$ we get $\frac{5}{6}$. For the other values of k we don't obtain solutions in the interval $[-1, 1]$. Consequently, the solutions to the problem are $-\frac{2}{3}$, $\frac{1}{3}$, $-\frac{1}{6}$, $-\frac{2}{3}$.

(13) Find all solutions of the equation in $[-1, 1]$.

(9) $$\sin(2\pi x) + \sqrt{3}\cos(2\pi x) = 1$$

Solution: This problem looks very similar to the one above but is solved in a different way. Define the function

$$f(x) = \sin(2\pi x) + \sqrt{3}\cos(2\pi x)$$

We have to write this function in the form

$$f(x) = A\sin(2\pi x + \phi)$$

Using the sum-of-angles formula we have:

$$A\sin(2\pi x + \phi) == A\sin(2\pi x)\cos\phi + A\cos(2\pi x)\sin\phi$$

Thus in order for the two formulas for $f(x)$ to be the same we must have

$$A\cos\phi = 1 \qquad\qquad A\sin\phi = \sqrt{3}$$

Squaring both sides of both equations we have

$$A^2\cos^2\phi = 1 \qquad\qquad A^2\sin^2\phi = 3$$

and adding them we have

$$A^2(\cos^2\phi + \sin^2\phi) = 4$$

or $A^2 = 4$, or $A = 2$. The angle ϕ has to satisfy

$$\cos\phi = \frac{1}{2} \qquad\qquad \sin\phi = \frac{\sqrt{3}}{2}.$$

Thus $\phi = \frac{\pi}{3}$. Consequently,

$$f(x) = 2\sin(2\pi x + \frac{\pi}{3})$$

Thus, we are solving the equation

$$2\sin(2\pi x + \frac{\pi}{3}) = 1$$

or

$$\sin\theta = \frac{1}{2}$$

All solutions to this equation are

$$\theta_1 = \frac{\pi}{6} + k\cdot 2\pi \qquad\qquad \theta_2 = \frac{5\pi}{6} + k\cdot 2\pi$$

From the first set of solutions we have

$$2\pi x_1 + \frac{\pi}{3} = \frac{\pi}{6} + k\cdot 2\pi$$
$$2\pi x_1 = -\frac{\pi}{6} + k\cdot 2\pi$$

Dividing by 2π

$$x_1 = -\frac{1}{12} + k$$

For $k = 0$ we get $-\frac{1}{12}$. For $k = 1$ we get $\frac{11}{12}$. For the other values of k we don't obtain solutions in the interval $[-1, 1]$. From the second set of solutions for θ we get

$$\theta_2 = \frac{5\pi}{6} + k\cdot 2\pi$$
$$2\pi x_2 + \frac{\pi}{3} = \frac{5\pi}{6} + k\cdot 2\pi$$
$$2\pi x_2 = \frac{\pi}{2} + k\cdot 2\pi$$
$$x_2 = \frac{1}{4} + k$$

If $k = -1$ we get $-\frac{3}{4}$. If $k = 0$ we get $\frac{1}{4}$ For the other values of k we don't obtain solutions in the interval $[-1, 1]$. Consequently, all solutions of the problem are $-\frac{1}{12}$, $\frac{11}{12}$, $-\frac{3}{4}$, $\frac{1}{4}$.

(14) A person's blood pressure, P, measured in millimeters of mercury (abbreviated as mm Hg), is given by

$$P = 100 - 20\cos\left(\frac{8\pi}{3}t\right), \tag{10}$$

where t is the time in seconds.

(a) Sketch the graph of this function.

(b) State the period, the amplitude and the midline. Explain their practical significance.

(c) For how long is the blood pressure above 110 mm Hg during the first second?

Solution:

(a)

(b) The midline is 100 and it gives the average blood pressure. The amplitude is 20 and it gives the maximal deviation from the average blood pressure. The period is

$$\text{period} = \frac{2\pi}{\frac{8\pi}{3}}$$

Thus the period is $\frac{3}{4}$. The period gives the time in which the blood pressure completes one cycle.

(c) We want to know when the blood pressure is 110 mm Hg. From the figure above it can be seen that if the blood pressure is 110 mm Hg at times t_1 and t_2 then it is above 110 mm Hg for the period of time between t_1 and t_2. Thus the total time for which the blood pressure is above 110 mm Hg is $t_2 - t_1$. Thus, the problem reduces to finding the solution of the equation

$$P(t) = 110$$

in the interval $[0, 1)$. The equation becomes

$$100 - 20\cos\left(\frac{8\pi}{3}t\right) = 110$$

Solving for the cosine we have

$$-20\cos\left(\frac{8\pi}{3}t\right) = 10$$

$$\cos\left(\frac{8\pi}{3}t\right) = -\frac{1}{2}$$

We set $\theta = \frac{8\pi}{3}t$. All solutions of the equation

$$\cos\theta = -\frac{1}{2}$$

are

$$\theta_1 = \frac{2\pi}{3} + k\cdot 2\pi \qquad \theta_2 = \frac{4\pi}{3} + k\cdot 2\pi$$

where $k = 0, \pm1, \pm2, \ldots$. From the first set of solutions for θ we have

$$\frac{8\pi}{3}t_1 = \frac{2\pi}{3} + k\cdot 2\pi$$

$$t_1 = \frac{1}{4} + k\cdot\frac{3}{4}$$

from which we obtain $t_1 = \frac{1}{4}$. From the second set of solutions for θ we have

$$\frac{8\pi}{3}t_2 = \frac{4\pi}{3} + k\cdot 2\pi$$

$$t_2 = \frac{1}{2} + k\cdot\frac{3}{4}$$

from which we obtain $t_2 = \frac{1}{2}$. Thus the amount of time during the first second for which the blood pressure is above 110 mm Hg is $\frac{1}{4}$ seconds.

(15) A ferris wheel has a radius of 10 meters, and the bottom of the wheel is 2 meters above the ground. Boarding is done from a platform at the 6 o'clock position. The ferris wheel makes one complete revolution every 20 minutes.

(a) Graph the hight above the ground of a person on the ferris wheel as a function of the time, t, after boarding.

(b) Find a formula for the hight above the ground, $h(t)$, of a person on the ferris wheel as a a function of the time, t, after boarding.

(c) How much time in one full revolution does a passenger spend above 17 meters?

Solution:

(a)

(b) We are looking for a sinusoidal function in terms of sine or cosine. Using cosine here is more convenient (there will be no shift) since the graph at time zero is at a minimal point and it looks like a reflected cosine function. (If the graph at time zero starts from the midline, then sine function will be more convenient.) Thus, we look for a function of the form

$$h(t) = A\cos Bt + k$$

Since the function oscillates between a minimum of 2 and a maximum of 22 then the average is 12 ($\frac{22+2}{2}$). Thus, the midline is $y = 12$ and $k = 12$. The amplitude is equal to the radius of the wheel, that is 10. Since the graph is a reflected cosine, then $A = -10$. The period is 20, thus

$$B = \frac{2\pi}{\text{period}} = \frac{\pi}{10}$$

Consequently, we obtain the formula

$$h(t) = -10\cos\left(\frac{\pi}{10}t\right) + 12.$$

(c) If t_1 and t_2 are the times when the passenger is at 17 meters above the ground, then he/she spends $t_2 - t_1$ time units above 17 meters above the ground (see the graph of $h(t)$). Thus, we have to find the solutions t_1 and t_2 of the equation

$$h(t) = 17$$

in the interval $[0, 20]$. We get

$$-10\cos\left(\frac{\pi}{10}t\right) + 12 = 17$$

Solving for the cosine

$$-10\cos\left(\frac{\pi}{10}t\right) = 5$$

$$\cos\left(\frac{\pi}{10}t\right) = -\frac{1}{2}$$

Let $\theta = \left(\frac{\pi}{10}t\right)$. We find all solutions of the equation

$$\cos\theta = -\frac{1}{2}.$$

These are

$$\theta_1 = \frac{2\pi}{3} + k \cdot 2\pi \qquad \theta_2 = \frac{4\pi}{3} + k \cdot 2\pi$$

where $k = 0, \pm 1, \pm 2, \ldots$. From the first set of solutions for θ we obtain

$$\frac{\pi}{10}t_1 = \frac{2\pi}{3} + k \cdot 2\pi$$

Solving for t we have

$$t_1 = \frac{20}{3} + 20k$$

We obtain the only solution in the interval $[0, 20]$ for $k = 0$ and it is $t_1 = \frac{20}{3}$. From the second set of solutions for θ we have

$$\frac{\pi}{10}t_2 = \frac{4\pi}{3} + k \cdot 2\pi$$

Solving for t we have

$$t_2 = \frac{40}{3} + 20k$$

We obtain the only solution in the interval $[0, 20]$ for $k = 0$ and it is $t_2 = \frac{40}{3}$. Thus the passenger spends $t_2 - t_1 = \frac{20}{3}$ minutes above 17 meters above the ground.

(16) The number of cases of the infectious disease dengue fever in French Polynesia oscillates sinusoidally during the year from a high of 200 cases on February 1 to a low of 20 cases on August 1.

(a) Graph the number of cases of dengue fever as a function of time for a period of one year starting on January 1.

(b) Find a formula for the number of cases, $N(t)$, as a function of time, t, in months since the start of the year.

(c) During which months of the year is the number of cases below 65?

Solution:

(a)

(b) We look for a function of the form

$$N(t) = A \sin B(t - c) + k$$

Since the number of cases oscillates between 20 and 200, that means the amplitude of this sinusoidal function is half of $200 - 20$ or 90. Looking at the graph above we can think of it as a sine wave, shifted 2 units to the left. Hence $A = 2$ and $c = -2$. The midline is 90 units above the minimum, or $90 + 20 = 110$. Hence the line $y = 110$ is a midline. From that it follows that $k = 110$. The period is 12. Since

$$\text{period} = \frac{2\pi}{B}$$

from here we get that $B = \frac{\pi}{6}$. Consequently, we get the formula

$$N(t) = 90 \sin\left(\frac{\pi}{6}(t+2)\right) + 110.$$

(c) Looking at the graph above we see that the two solutions t_1 and t_2 of the equation $N(t) = 65$ will give us correspondingly the beginning of this period and the end of the period. We have to solve the equation

$$90 \sin\left(\frac{\pi}{6}(t+2)\right) + 110 = 65$$

in the interval $[0, 12]$. First we solve for the sine:

$$90 \sin\left(\frac{\pi}{6}(t+2)\right) = -45$$

$$\sin\left(\frac{\pi}{6}(t+2)\right) = -0.5$$

Let $\theta = \frac{\pi}{6}(t+2)$. We first find all solutions of the equation

$$\sin\theta = -0.5$$

which are

$$\theta_1 = \frac{7\pi}{6} + k \cdot 2\pi \qquad \theta_2 = \frac{11\pi}{6} + k \cdot 2\pi$$

where $k = 0, \pm 1, \pm 2, \ldots$. From the first solution for θ we get

$$\frac{\pi}{6}(t+2) = \frac{7\pi}{6} + k \cdot 2\pi$$

$$t + 2 = 7 + 12k$$

We obtain the only solution in the interval $[0, 12]$ for $k = 0$ and it is $t_1 = 5$. From the second set of solutions for θ we have

$$\frac{\pi}{6}(t + 2) = \frac{11\pi}{6} + k \cdot 2\pi$$
$$t + 2 = 11 + 12k$$

We obtain the only solution in the interval $[0, 12]$ for $k = 0$ and it is $t_2 = 9$. Thus the number of dangue fever cases is below 65 between June 1 and October 1.

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