Polytechnic University

MA 1122/1132 **SAMPLE FINAL** FALL, 2006

MIDTERM: ______

FINAL: ______

GRADE: ______

Print Name:

Signature:

ID #:

Instructor/Section:

Directions: You have **90 minutes** to answer the following questions. ***You must show all your work*** as neatly and clearly as possible and indicate the final answer clearly. You may use only a TI-30 calculator. The last few pages contain formulas that you might find useful. You may tear those pages out.

If you are feeling ill you should inform the proctor. The proctor will note your name, Poly ID and accept any written statement(s) that you may wish to make regarding your illness.

Problem	Possible	Points
1	10	
2	15	
3	10	
4	8	
5	12	
6	10	
7	10	
8	15	
9	10	
Total	100	

YOUR SIGNATURE: ____________________

(1) Decide if the following integrals converge. If so, find an upper bound for the value of the integral. You must explain clearly to receive credit.

(a) $\displaystyle\int_1^\infty \frac{3-\cos(\phi)}{\sqrt{\phi}}\,d\phi$

(b) $\displaystyle\int_0^{\pi/4} \frac{\cos(2x)dx}{(\sin(2x)+\sin^5(2x))^{1/3}}$

YOUR SIGNATURE: ____________________

(2) (Page 416, Problems 9–20; Page 422, Problems 21–28) Determine whether each of the following series converges. Circle all convergent series. You do not have to explain.

(a) $\sum_{n=0}^{\infty}(-1)^n \frac{n}{n+1}$

(b) $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$

(c) $\sum_{n=0}^{\infty} \frac{5}{n+1}$

(d) $\sum_{n=1}^{\infty} \frac{2^n}{n!}$

(e) $\sum_{n=0}^{\infty} \frac{3^{n+5}}{5^n}$

(3) (Page 428, Problem 21) Find the radius of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(x-5)^{3n}}{2 \cdot 4^n}.$$

Show all your work.

YOUR SIGNATURE:

(4) (Page 445, Problem 21) Find the Taylor series for

$$f(x) = \sin(2x - \pi)$$

for x near π up to the first two non-zero terms. Show all your work.

YOUR SIGNATURE:

(5) (Page 446, Problems 36–37) Using known Taylor series, solve **exactly** for the variable x. Show all your work.

$$-x^2 - \frac{x^3}{2} - \frac{x^4}{3} - \frac{x^5}{4} - \cdots - \frac{x^{n+1}}{n} - \cdots = \frac{x}{3}$$

(6) OLD(Concepts) Determine whether each of the following statements is TRUE or FALSE. You do not have to explain.

(a) The Taylor series of $f(x)g(x)$ near 0 is

$$f(0)g(0) + f'(0)g'(0)x + \frac{f''(0)g''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)g^{(n)}(0)}{n!}x^n + \cdots .$$

(b) If the Taylor polynomial of degree 2 for $f(x)$ near 0 is $T_2(x) = 2x - 3x^2$, then $f(x)$ is increasing near $x = 0$.

(c) If the Taylor polynomial of degree 2 for $g(x)$ near 0 is $T_2(x) = 1 - 3x - x^2$, then $g(x)$ has a local maximum at $x = 0$.

(d) The Taylor series of $h(x) = \dfrac{1}{1-x}$ near $x = 5$ is

$$1 + (x-5) + (x-5)^2 + (x-5)^3 + \cdots + (x-5)^n + \cdots .$$

(e) If $f(x)$ and $g(x)$ have the same Taylor polynomial of degree 2 near $x = 0$, then $f(x) = g(x)$.

Your signature:

(7) (Page 441, Problem 26) Use Taylor series for h near 0 to evaluate the following limit. Show all your work.

$$\lim_{h \to 0} \frac{\cos(2h) - 1}{3h - 1 + e^{-3h}}$$

(8) (Page 368, Problem 22) Let $\mathcal{R}$ be a plate bounded by the two curves $y = 4x - x^3$, $y = x^3 - 4x$, and the two lines $x = 0$ and $x = 2$. Suppose that a plate of uniform density 3 gm/cm^2, with x and y measured in cm.

(a) Sketch the region $\mathcal{R}$.

(b) Find the total mass of the plate. Show all your work.

(c) Find its center of mass, $(\bar{x}, \bar{y})$. Show all your work.

(9) (Sample Exam) The sixth degree Taylor polynomial of a function f about $x = 5$ is:

$$3 - \frac{1}{3}(x-5)^2 + \frac{1}{120}(x-5)^4 - \frac{7}{30}(x-5)^6.$$

For each of the statements below, decide whether it is TRUE or FALSE or state if there is NEI, NOT ENOUGH INFORMATION to determine. You do not have to show work.

(a) $f(5) = 3$

(b) $f'(5) = -\frac{1}{3}$

(c) $f''(5) = -\frac{2}{3}$

(d) $f^{(6)}(5) = -168$

(e) f has a local maximum at $x = 5$.

Useful formulas

- *Geometry Formulas*
  Here V is the volume, S is the surface area, h is the height and r is the radius.
  Cylinder with top and bottom: $V = \pi r^2 h$, $S = 2\pi rh + 2\pi r^2$
  Cone: $V = \frac{1}{3}\pi r^2 h$
  Sphere: $V = \frac{4}{3}\pi r^3$, $S = 4\pi r^2$
- *Physics formulas:*
  The *acceleration* due to gravity, g: $g = 9.8\text{m/sec}^2$, or $g = 32\text{ft/sec}^2$.
  Mass density of water = 1000 kg/m^3, Weight density of water = 62.4 lbs/ft^3.
  Force = mass × acceleration Work = Force × distance
  The center of mass, $\bar{x}$, of an object lying on the x-axis between $x = a$ and $x = b$, with mass density $\delta(x)$ is given by $\bar{x} = \dfrac{\int_a^b x\delta(x)\,dx}{\text{total mass}}$

  Arc length of a curve $y = f(x)$ from $x = a$ to $x = b$: $L = \displaystyle\int_a^b \sqrt{1 + (f'(x))^2}\,dx$
- *Integration by Parts*:
  $$\int u dv = uv - \int v du \qquad \text{or} \qquad \int uv' dx = uv - \int vu' dx$$
- *Numerical Approximations:*

  $\text{TRAP}(n) = \frac{\text{LEFT}(n)+\text{RIGHT}(n)}{2}$; $\text{SIMP}(n) = \frac{2\text{MID}(n)+\text{TRAP}(n)}{3}$

- Finite Geometric Sum:
  $$a + ax + ax^2 + \cdots + ax^{n-1} = \frac{a(1 - x^n)}{1 - x}$$
- Infinite Geometric Series:
  $$a + ax + ax^2 + \cdots = \frac{a}{1 - x} \qquad \text{for } |x| < 1$$

- Ratio Test:
  For the series $\sum a_n$, suppose,
  $$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L.$$
  - If $L < 1$, then the series converges.
  - If $L > 1$, then the series diverges.
  - If $L = 1$, then the test fails.

- *Useful Integrals for Comparison:*

  $\int_1^\infty \frac{1}{x^p}\,dx$ converges for $p > 1$ and diverges for $p \leq 1$.

  $\int_0^1 \frac{1}{x^p}\,dx$ converges for $p < 1$ and diverges for $p \geq 1$.

  $\int_0^\infty e^{-ax}\,dx$ converges for $a > 0$.

- nth degree Taylor Polynomial of $f(x)$ centered at $x = a$:

  $f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$

- Taylor series of $f(x)$ centered at $x = a$:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

- Taylor Series of important functions centered at $x = 0$:

  $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

  $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

  $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

  $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$ for $-1 < x < 1$

  $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ for $-1 < x \leq 1$

  $(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots$ for $-1 < x < 1$

- **Differentiation formulas**

$\frac{d}{dx}(x^n) = nx^{n-1}$ $\frac{d}{dx}(\ln\lvert x\rvert) = \frac{1}{x}$	$\frac{d}{dx}(e^x) = e^x$ $\frac{d}{dx}(\sin(x)) = \cos x$	$\frac{d}{dx}(a^x) = (\ln a)a^x$ $\frac{d}{dx}(\cos(x)) = -\sin x$
	$\frac{d}{dx}(\tan(x)) = \sec^2 x$	$\frac{d}{dx}(\cot(x)) = -\csc^2 x$
	$\frac{d}{dx}(\sec(x)) = \sec x \tan x$	$\frac{d}{dx}(\csc(x)) = -\csc x \cot x$
$\frac{d}{dx}(\arcsin(x)) = \frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}(\arccos(x)) = \frac{-1}{\sqrt{1-x^2}}$	$\frac{d}{dx}(\arctan(x)) = \frac{1}{1+x^2}$

YOUR SIGNATURE: ____________________

Here a, b, c, d are constants.

A Short Table of Indefinite Integrals

I. Basic Functions

1. $\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C, \; (n \neq -1)$
2. $\int \frac{1}{x} \, dx = \ln|x| + C$
3. $\int a^x \, dx = \frac{1}{\ln a} a^x + C$
4. $\int \ln x \, dx = x \ln x - x + C$
5. $\int \sin ax \, dx = -\frac{1}{a} \cos ax + C$
6. $\int \cos ax \, dx = \frac{1}{a} \sin ax + C$
7. $\int \tan ax \, dx = -\frac{1}{a} \ln|\cos ax| + C$

II. Products of e^x, $\cos x$, and $\sin x$

8. $\int e^{ax} \sin(bx) \, dx = \frac{1}{a^2+b^2} e^{ax}[a\sin(bx) - b\cos(bx)] + C$
9. $\int e^{ax} \cos(bx) \, dx = \frac{1}{a^2+b^2} e^{ax}[a\cos(bx) + b\sin(bx)] + C$
10. $\int \sin(ax)\sin(bx) \, dx = \frac{1}{b^2-a^2}[a\cos(ax)\sin(bx) - b\sin(ax)\cos(bx)] + C, \quad a \neq b$
11. $\int \cos(ax)\cos(bx) \, dx = \frac{1}{b^2-a^2}[b\cos(ax)\sin(bx) - a\sin(ax)\cos(bx)] + C, \quad a \neq b$
12. $\int \sin(ax)\cos(bx) \, dx = \frac{1}{b^2-a^2}[b\sin(ax)\sin(bx) + a\cos(ax)\cos(bx)] + C, \quad a \neq b$

III. Product of Polynomial $p(x)$ with $\ln x$, e^x, $\cos x$, and $\sin x$

13. $\int x^n \ln x \, dx = \frac{1}{n+1} x^{n+1} \ln x - \frac{1}{(n+1)^2} x^{n+1} + C, \quad n \neq -1, x > 0$
14. $\int p(x) e^{ax} \, dx = \frac{1}{a} p(x) e^{ax} - \frac{1}{a^2} p'(x) e^{ax} + \frac{1}{a^3} p''(x) e^{ax} - \cdots + C$

    $(+ - + - + - + \ldots)$ (signs alternate)
15. $\int p(x) \sin ax \, dx = -\frac{1}{a} p(x)\cos(ax) + \frac{1}{a^2} p'(x)\sin(ax) + \frac{1}{a^3} p''(x)\cos(ax) - \cdots + C$

    $(- + + - - + + - - \ldots)$ (signs alternate in pairs)
16. $\int p(x) \cos ax \, dx = \frac{1}{a} p(x)\sin(ax) + \frac{1}{a^2} p'(x)\cos(ax) - \frac{1}{a^3} p''(x)\sin(ax) - \cdots + C$

    $(+ + - - + + - - \ldots)$ (signs alternate in pairs)

IV. Integer Powers of $\sin x$ and $\cos x$

17. $\displaystyle\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1} x\cos x + \frac{n-1}{n}\int \sin^{n-2} x\,dx, \quad n \text{ positive}$

18. $\displaystyle\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x\sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx, \quad n \text{ positive}$

19. $\displaystyle\int \frac{1}{\sin^m x}\,dx = -\frac{1}{m-1}\frac{\cos x}{\sin^{m-1} x} + \frac{m-2}{m-1}\int \frac{1}{\sin^{m-2} x}\,dx, \quad m \neq 1, m \text{ positive}$

20. $\displaystyle\int \frac{1}{\sin x}\,dx = \frac{1}{2}\ln\left|\frac{\cos x - 1}{\cos x + 1}\right| + C$

21. $\displaystyle\int \frac{1}{\cos^m x}\,dx = \frac{1}{m-1}\frac{\sin x}{\cos^{m-1} x} + \frac{m-2}{m-1}\int \frac{1}{\cos^{m-2} x}\,dx, \quad m \neq 1, m \text{ positive}$

22. $\displaystyle\int \frac{1}{\cos x}\,dx = \frac{1}{2}\ln\left|\frac{\sin x + 1}{\sin x - 1}\right| + C$

23. $\displaystyle\int \sin^m x\cos^n x\,dx :$

If n is odd, let $w = \sin x$.
If both m and n are even and non-negative, convert all to $\sin x$ or all to $\cos x$ (using $\sin^2 x + \cos^2 x = 1$), and use IV-17 or IV-18.
If m and n are even and one of them is negative, convert to whichever function is in the denominator and use IV-19 or IV-21.
The case in which both m and n are even and negative is omitted.

V. Quadratic in the Denominator

24. $\displaystyle\int \frac{1}{x^2 + a^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C,\ a \neq 0$

25. $\displaystyle\int \frac{bx + c}{x^2 + a^2}\,dx = \frac{b}{2}\ln|x^2 + a^2| + \frac{c}{a}\arctan\left(\frac{x}{a}\right) + C,\ a \neq 0$

26. $\displaystyle\int \frac{1}{(x-a)(x-b)}\,dx = \frac{1}{(a-b)}(\ln|x-a| - \ln|x-b|) + C, \quad a \neq b$

27. $\displaystyle\int \frac{cx + d}{(x-a)(x-b)}\,dx = \frac{1}{(a-b)}[(ac+d)\ln|x-a| - (bc+d)\ln|x-b|] + C, \quad a \neq b$

VI. Integrands involving $\sqrt{a^2+x^2}, \sqrt{a^2-x^2}, \sqrt{x^2-a^2}, a > 0$

28. $\displaystyle\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\left(\frac{x}{a}\right) + C$

29. $\displaystyle\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln|x + \sqrt{x^2 \pm a^2}| + C$

30. $\displaystyle\int \sqrt{a^2 \pm x^2}\,dx = \frac{1}{2}\left(x\sqrt{a^2 \pm x^2} + a^2\int \frac{1}{\sqrt{a^2 \pm x^2}}\,dx\right) + C$

31. $\displaystyle\int \sqrt{x^2 - a^2}\,dx = \frac{1}{2}\left(x\sqrt{x^2 - a^2} - a^2\int \frac{1}{\sqrt{x^2 - a^2}}\,dx\right) + C$