Polytechnic University

MA 1122/1422 Sample Exam 2 Spring 2005

Exam 1: ______

Exam 2: ______

Grade: ______

Print Name:

Signature:

ID #:

Instructor/Section:

Directions: You have **90 minutes** to answer the following questions. ***You must show all your work*** as neatly and clearly as possible and indicate the final answer clearly. You may use only a TI-30 calculator. The last few pages contain formulas that you might find useful. You may tear those pages out.

If you are feeling ill you should inform the proctor. The proctor will note your name, Poly ID and accept any written statement(s) that you may wish to make regarding your illness.

Problem	Possible	Points
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total	100	

(1) (Page 429, Problem 18) You are saving for a big trip abroad. You estimate that you'll need \$4000 to cover all of your expenses. You plan to put away a fixed amount of money every month for the next 2 years (24 deposits) so that immediately after the 24th deposit you have enough money for your trip. You put the money into an account paying an interest of 4.5% per year **compounded monthly.** How much money must you deposit every month? Show all your work.

(2) (Chapter 9, Concepts) Decide whether each of the statements is TRUE or FALSE. You do not have to explain.

(a) If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges.

(b) If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

(c) If $\sum_{n=1}^{\infty} |\, a_n|$ converges, then so does $\sum_{n=1}^{\infty} a_n$.

(d) If $a_n > 0$ and $\sum_{n=1}^{\infty} a_n$ converges, then so does $\sum_{n=1}^{\infty} (a_n)^n$.

(e) If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge, then both $\sum_{n=1}^{\infty} (a_n + b_n)$ and $\sum_{n=1}^{\infty} (a_n - b_n)$ converge.

(3) (Page 429, Problems 5–11) Determine if the following series converge. You may have to use the Ratio test, the Alternating series test, the Integral test, or recognize a geometric series. Circle the ones which converge. You do not need to show work.

(a) $\sum_{n=1}^{\infty} 3^n 2^{-n}$

(b) $\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^{3/2}}$

(c) $\sum_{n=0}^{\infty} \frac{3}{(n+2)!}$

(d) $\sum_{n=1}^{\infty} \frac{\cos(3n\pi)}{2n}$

(4) (Page 428, Problem 23) In each of the following parts, circle the **ONE** alternative that correctly completes the sentence. You do not need to explain.

(a) The radius of convergence of $\sum_{n=1}^{\infty} \frac{4^n(x-1)^{2n}}{n}$ is

(i) 4.

(ii) $\frac{1}{4}$.

(iii) 2.

(iv) $\frac{1}{2}$.

(v) None of the above.

(b) The radius of convergence of $\sum_{n=1}^{\infty} \frac{n!x^{2n}}{25^n}$ is

(i) 0.

(ii) 5.

(iii) 25.

(iv) ∞.

(v) None of the above.

(c) The interval of convergence of $\sum_{n=1}^{\infty} \frac{(2x)^n}{n}$ is

(i) $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

(ii) $\left[-\frac{1}{2}, \frac{1}{2}\right)$.

(iii) $\left(-\frac{1}{2}, \frac{1}{2}\right]$.

(iv) $\left[-\frac{1}{2}, \frac{1}{2}\right]$.

(v) None of the above.

(5) (Page 430, Problem 24) Suppose that the power series $\sum_{n=1}^{\infty} c_n(x-2)^n$ converges for $x = -10$ and diverges for $x = 20$.

For each of the statements below state whether it is TRUE, FALSE or NEI (there is not enough information to conclude one way or the other). You do not have to explain.

(a) The power series converges when $x = 10$.

(b) The power series converges when $x = -20$.

(c) The power series converges when $x = 14$.

(d) The power series converges when $x = 3$.

(e) The radius of convergence for this power series is 12.

(6) (Page 440, Problem 20) The table below gives some values of a function f and its derivatives.

x	$f(x)$	$f'(x)$	$f''(x)$	$f'''(x)$
0	2	1	4	7
2	0	-1.4	-3	-8
4	-2.4	4.8	-9	-2.1
8	-3.5	6.5	-3.7	-2

Suppose $g(x) = \displaystyle\int_{x/2}^{1} f(4t)dt$. Write the third degree Taylor polynomial for $g(x)$ about $x = 2$. You must show all your work.

(7) (Page 446, Problems 38 and 40) Suppose you know that all derivatives of some function f exist at 0, and that the Taylor series for f about $x = 0$ is

$$-1 + \frac{x^2}{2} - \frac{x^4}{4} + \frac{x^6}{6} - \frac{x^8}{8} + \frac{x^{10}}{10} - ...$$

Fill in the blanks. You do not have to show your work.

(a) The function has a ____________________ (local max or local min) at $x = 0$, the value of this local extremum is ____________________.

(b) $f^{(101)}(0) =$ ____________________

(c) $f^{(404)}(0) =$ ____________________

(d) If we use the second degree Taylor polynomial to estimate the value of $f(0.01)$, then the approximate value is

____________________.

(8) (Page 445, Problem 20) Use the first 3 nonzero terms of the Taylor series of $h(t) = \sin(t^2)$ near $t = 0$ to approximate the value of the definite integral

$$\int_0^{0.1} \sin(t^2)dt.$$

Show all your work.

(9) (Page 446, Problems 36–37) Use known Taylor series to solve the following equation for the exact value(s) of x.

$$x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \cdots = \ln(2)$$

Show your work.

(10) (Page 441, Problems 24–27) Use Taylor series to evaluate the limit

$$\lim_{t\to 0} \frac{t\ln(1+t)}{\sin(t^2)}.$$

Show your work clearly.

Useful formulas

- *Geometry Formulas*
  Here V is the volume, S is the surface area, h is the height and r is the radius.
  Cylinder with top and bottom: $V = \pi r^2 h$, $S = 2\pi rh + 2\pi r^2$
  Cone: $V = \frac{1}{3}\pi r^2 h$
  Sphere: $V = \frac{4}{3}\pi r^3$, $S = 4\pi r^2$
- *Physics formulas:*
  The *acceleration* due to gravity, g: $g = 9.8\text{m}/\text{sec}^2$, or $g = 32\text{ft}/\text{sec}^2$.
  Mass density of water = 1000 kg/m^3, Weight density of water = 62.4 lbs/ft^3.
  Force = mass × acceleration Work = Force × distance
  The center of mass, $\bar{x}$, of an object lying on the x-axis between $x = a$ and $x = b$, with mass density $\delta(x)$ is given by $\bar{x} = \dfrac{\int_a^b x\delta(x)\,dx}{\text{total mass}}$
  Arc length of a curve $y = f(x)$ from $x = a$ to $x = b$: $L = \displaystyle\int_a^b \sqrt{1 + (f'(x))^2}\,dx$
- *Integration by Parts*:
  $$\int u\,dv = uv - \int v\,du \qquad \text{or} \qquad \int uv'dx = uv - \int vu'dx$$
- *Numerical Approximations:*
  $$\text{TRAP}(n) = \frac{\text{LEFT}(n)+\text{RIGHT}(n)}{2}; \qquad \text{SIMP}(n) = \frac{2\text{MID}(n)+\text{TRAP}(n)}{3}$$
- Finite Geometric Sum:
  $$a + ax + ax^2 + \cdots + ax^{n-1} = \frac{a(1-x^n)}{1-x}$$
- Infinite Geometric Series:
  $$a + ax + ax^2 + \cdots = \frac{a}{1-x} \qquad \text{for } |x| < 1$$
- Ratio Test:
  For the series $\sum a_n$, suppose,
  $$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L.$$
  - If $L < 1$, then the series converges.
  - If $L > 1$, then the series diverges.
  - If $L = 1$, then the test fails.

- nth degree Taylor Polynomial of $f(x)$ centered at $x = a$:

  $f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$
- Taylor series of $f(x)$ centered at $x = a$:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

- Taylor Series of important functions:

  $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

  $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

  $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

  $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \qquad \text{for } -1 < x < 1$

  $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad \text{for } -1 < x \le 1$

  $(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots \qquad \text{for } -1 < x < 1$
- **Differentiation formulas**

$\frac{d}{dx}(x^n) = nx^{n-1}$	$\frac{d}{dx}(e^x) = e^x$	$\frac{d}{dx}(a^x) = (\ln a)a^x$
$\frac{d}{dx}(\ln\|x\|) = \frac{1}{x}$	$\frac{d}{dx}(\sin(x)) = \cos x$	$\frac{d}{dx}(\cos(x)) = -\sin x$
	$\frac{d}{dx}(\tan(x)) = \sec^2 x$	$\frac{d}{dx}(\cot(x)) = -\csc^2 x$
	$\frac{d}{dx}(\sec(x)) = \sec x \tan x$	$\frac{d}{dx}(\csc(x)) = -\csc x \cot x$
$\frac{d}{dx}(\arcsin(x)) = \frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}(\arccos(x)) = \frac{-1}{\sqrt{1-x^2}}$	$\frac{d}{dx}(\arctan(x)) = \frac{1}{1+x^2}$

Here a, b, c, d are constants.

A Short Table of Indefinite Integrals

I. Basic Functions

1. $\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C$, $(n \neq -1)$
2. $\int \frac{1}{x} \, dx = \ln|x| + C$
3. $\int a^x \, dx = \frac{1}{\ln a} a^x + C$
4. $\int \ln x \, dx = x \ln x - x + C$
5. $\int \sin ax \, dx = -\frac{1}{a} \cos ax + C$
6. $\int \cos ax \, dx = \frac{1}{a} \sin ax + C$
7. $\int \tan ax \, dx = -\frac{1}{a} \ln|\cos ax| + C$

II. Products of e^x, $\cos x$, and $\sin x$

8. $\int e^{ax} \sin(bx) \, dx = \frac{1}{a^2 + b^2} e^{ax}[a \sin(bx) - b \cos(bx)] + C$
9. $\int e^{ax} \cos(bx) \, dx = \frac{1}{a^2 + b^2} e^{ax}[a \cos(bx) + b \sin(bx)] + C$
10. $\int \sin(ax) \sin(bx) \, dx = \frac{1}{b^2 - a^2}[a \cos(ax) \sin(bx) - b \sin(ax) \cos(bx)] + C, \quad a \neq b$
11. $\int \cos(ax) \cos(bx) \, dx = \frac{1}{b^2 - a^2}[b \cos(ax) \sin(bx) - a \sin(ax) \cos(bx)] + C, \quad a \neq b$
12. $\int \sin(ax) \cos(bx) \, dx = \frac{1}{b^2 - a^2}[b \sin(ax) \sin(bx) + a \cos(ax) \cos(bx)] + C, \quad a \neq b$

III. Product of Polynomial $p(x)$ with $\ln x$,e^x, $\cos x$, and $\sin x$

13. $\int x^n \ln x \, dx = \frac{1}{n+1} x^{n+1} \ln x - \frac{1}{(n+1)^2} x^{n+1} + C, \quad n \neq -1, x > 0$
14. $\int p(x) e^{ax} \, dx = \frac{1}{a} p(x) e^{ax} - \frac{1}{a^2} p'(x) e^{ax} + \frac{1}{a^3} p''(x) e^{ax} - \cdots + C$

    $(+ - + - + - + \ldots)$ (signs alternate)
15. $\int p(x) \sin ax \, dx = -\frac{1}{a} p(x) \cos(ax) + \frac{1}{a^2} p'(x) \sin(ax) + \frac{1}{a^3} p''(x) \cos(ax) - \cdots + C$

    $(- + + - - + + - - \ldots)$ (signs alternate in pairs)
16. $\int p(x) \cos ax \, dx = \frac{1}{a} p(x) \sin(ax) + \frac{1}{a^2} p'(x) \cos(ax) - \frac{1}{a^3} p''(x) \sin(ax) - \cdots + C$

    $(+ + - - + + - - \ldots)$ (signs alternate in pairs)

IV. Integer Powers of $\sin x$ and $\cos x$

17. $\displaystyle\int \sin^n x\ dx = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x\ dx, \quad n \text{ positive}$

18. $\displaystyle\int \cos^n x\ dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x\ dx, \quad n \text{ positive}$

19. $\displaystyle\int \frac{1}{\sin^m x}\ dx = -\frac{1}{m-1}\frac{\cos x}{\sin^{m-1} x} + \frac{m-2}{m-1}\int \frac{1}{\sin^{m-2} x}\ dx, \quad m \neq 1, m \text{ positive}$

20. $\displaystyle\int \frac{1}{\sin x}\ dx = \frac{1}{2}\ln\left|\frac{\cos x - 1}{\cos x + 1}\right| + C$

21. $\displaystyle\int \frac{1}{\cos^m x}\ dx = \frac{1}{m-1}\frac{\sin x}{\cos^{m-1} x} + \frac{m-2}{m-1}\int \frac{1}{\cos^{m-2} x}\ dx, \quad m \neq 1, m \text{ positive}$

22. $\displaystyle\int \frac{1}{\cos x}\ dx = \frac{1}{2}\ln\left|\frac{\sin x + 1}{\sin x - 1}\right| + C$

23. $\displaystyle\int \sin^m x \cos^n x\ dx :$

If n is odd, let $w = \sin x$.
If both m and n are even and non-negative, convert all to $\sin x$ or all to $\cos x$ (using $\sin^2 x + \cos^2 x = 1$), and use IV-17 or IV-18.
If m and n are even and one of them is negative, convert to whichever function is in the denominator and use IV-19 or IV-21.
The case in which both m and n are even and negative is omitted.

V. Quadratic in the Denominator

24. $\displaystyle\int \frac{1}{x^2 + a^2}\ dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C,\ a \neq 0$

25. $\displaystyle\int \frac{bx + c}{x^2 + a^2}\ dx = \frac{b}{2}\ln|x^2 + a^2| + \frac{c}{a}\arctan\left(\frac{x}{a}\right) + C,\ a \neq 0$

26. $\displaystyle\int \frac{1}{(x-a)(x-b)}\ dx = \frac{1}{(a-b)}(\ln|x-a| - \ln|x-b|) + C, \quad a \neq b$

27. $\displaystyle\int \frac{cx + d}{(x-a)(x-b)}\ dx = \frac{1}{(a-b)}[(ac+d)\ln|x-a| - (bc+d)\ln|x-b|] + C, \quad a \neq$ b

VI. Integrands involving $\sqrt{a^2 + x^2}, \sqrt{a^2 - x^2}, \sqrt{x^2 - a^2}, a > 0$

28. $\displaystyle\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\left(\frac{x}{a}\right) + C$

29. $\displaystyle\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln|x + \sqrt{x^2 \pm a^2}| + C$

30. $\displaystyle\int \sqrt{a^2 \pm x^2}\ dx = \frac{1}{2}\left(x\sqrt{a^2 \pm x^2} + a^2\int \frac{1}{\sqrt{a^2 \pm x^2}}\ dx\right) + C$

31. $\displaystyle\int \sqrt{x^2 - a^2}\ dx = \frac{1}{2}\left(x\sqrt{x^2 - a^2} - a^2\int \frac{1}{\sqrt{x^2 - a^2}}\ dx\right) + C$