Polytechnic University

MA 1122 | FINAL | MAY 8, 2006

Print Name:

Signature:

ID #:

Instructor/Section:

Directions: You have **90 minutes** to answer the following questions. ***You must show all your work*** as neatly and clearly as possible and indicate the final answer clearly. You may use only a TI-30 calculator. The last few pages contain formulas that you might find useful. You may tear those pages out.

If you are feeling ill you should inform the proctor. The proctor will note your name, Poly ID and accept any written statement(s) that you may wish to make regarding your illness.

Problem	Possible	Points
1	15	
2	11	
3	12	
4	10	
5	8	
6	12	
7	10	
8	10	
9	12	
Total	100	

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(1) (Page 454, Problems 10–19; Page 462, Problems 42–53) Determine which of the series converges. Circle all that converge. You do not need to explain.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{e^{-n}}$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{4^n}{3^{2n}}$

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{(n-1)!}{2^n}$

(d) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n^2+1)}$

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(2) (Page 484, Problem 15) In each part, find the first three non-zero terms of the Taylor series for the function $f(x)$ about the point a. Show all your work.

(a) $f(x) = \sin(x^2), \quad a = 0$

(b) $f(x) = \ln(2x), \quad a = 1/3$

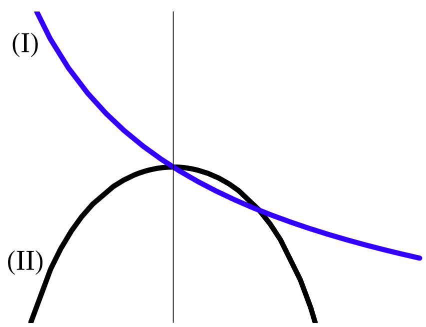
(3) (Page 485, Problem 36) The first few terms of the Taylor series of 4 functions near 0 is given in parts (a)–(d). Match each of the Taylor series (a)–(d) with the graphs in figures (I)–(IV) below. You do not need to explain.

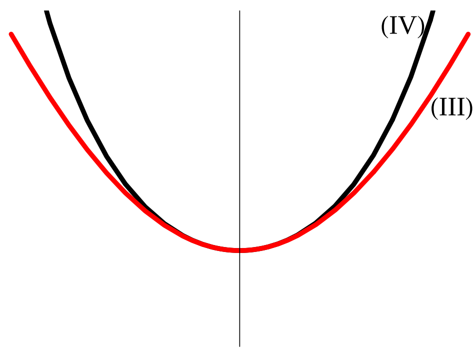
(a) $1 - x^2 - x^4 - x^6 - \cdots$

(b) $1 + x^2 - \frac{x^4}{6} + \frac{x^6}{120} - \cdots$

(c) $1 - \frac{x}{2} + \frac{3x^2}{8} - \frac{5x^3}{16} - \frac{35x^4}{128} - \cdots$

(d) $1 + x^2 + x^4 + \frac{x^6}{2} + \cdots$





Fill in the blanks.

Taylor series (a) corresponds to graph __________.

Taylor series (b) corresponds to graph __________.

Taylor series (c) corresponds to graph __________.

Taylor series (d) corresponds to graph __________.

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(4) (Page 470, Problem 26) Find the radius of convergence of the series

$$\sum_{n=1}^{\infty} \frac{n^2(x-5)^{2n}}{6^n}.$$

Show all your work.

(5) (Page 448, Problems 1–21) In each part, circle the correct choice. You need not show any work.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{5^n} =$

(i) $-2/3$.

(ii) $-2/7$.

(iii) $5/7$.

(iv) $5/3$.

(v) The series diverges.

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{3^{3n}}{5^{2n}} =$

(i) $3/5$.

(ii) $5/2$.

(iii) $5/8$.

(iv) $-25/2$.

(v) The series diverges.

(6) (Page 355, Problems 11 and 24) In each part, decide if the improper integral converges or diverges. If the integral converges, then find its **exact** value. Show all your work. If you use a rule from the tables of integrals, state the rule you are using.

(a) $\displaystyle\int_1^{e^8} \frac{1}{x\sqrt[3]{\ln x}}\,dx$

(b) $\displaystyle\int_0^{\infty} \frac{e^{2t}}{1+e^{2t}}\,dt$

(7) (Page 395, Example 7; Page 397, Problem 3) A 2-meter rod lies along the x-axis with its left end at the origin. At a distance x meters from its left end, the density of the rod is given by

$$\delta(x) = 2 + 6x \text{ g/m}.$$

(a) Find the total mass of the rod. Show all your work.

(b) Find the center of mass of the rod. Show all your work.

(8) (Page 380, Problems 19–24) The region $\mathcal{R}$ is bounded by $y = x^3$, $x = 3$ and $y = -8$. **Set up**, but do not evaluate, the integral which gives the volume of the solid obtained by rotating region $\mathcal{R}$ about the line $y = -8$. You must show all your work.

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(9) (Page 319, Problem 55; Page 330, Problem 31) In each part, find the integral. Show all your work. If you use a rule from the tables of integrals, state the rule you are using.

(a) $\displaystyle\int_{\pi/6}^{\pi/4} 3^{\cos(2\theta)} \sin(2\theta)\, d\theta$

(b) $\displaystyle\int \frac{1}{3z - z^2}\, dz$

Useful formulas

- *Geometry Formulas*
  Here V is the volume, S is the surface area, h is the height and r is the radius.
  Cylinder with top and bottom: $V = \pi r^2 h$, $S = 2\pi rh + 2\pi r^2$
  Cone: $V = \frac{1}{3}\pi r^2 h$
  Sphere: $V = \frac{4}{3}\pi r^3$, $S = 4\pi r^2$
- *Physics formulas:*
  The *acceleration* due to gravity, g: $g = 9.8\text{m/sec}^2$, or $g = 32\text{ft/sec}^2$.
  Mass density of water = 1000 kg/m^3, Weight density of water = 62.4 lbs/ft^3.
  Force = mass × acceleration Work = Force × distance
  The center of mass, $\bar{x}$, of an object lying on the x-axis between $x = a$ and $x = b$, with mass density $\delta(x)$ is given by $\bar{x} = \dfrac{\int_a^b x\delta(x)\,dx}{\text{total mass}}$

  Arc length of a curve $y = f(x)$ from $x = a$ to $x = b$: $L = \displaystyle\int_a^b \sqrt{1 + (f'(x))^2}\,dx$
- *Integration by Parts*:

$$\int u dv = uv - \int v du \qquad \text{or} \qquad \int uv' dx = uv - \int vu' dx$$

- *Numerical Approximations:*

$$\text{TRAP}(n) = \frac{\text{LEFT}(n) + \text{RIGHT}(n)}{2}; \qquad \text{SIMP}(n) = \frac{2\text{MID}(n) + \text{TRAP}(n)}{3}$$

- Finite Geometric Sum:

$$a + ax + ax^2 + \cdots + ax^{n-1} = \frac{a(1 - x^n)}{1 - x}$$

- Infinite Geometric Series:

$$a + ax + ax^2 + \cdots = \frac{a}{1 - x} \qquad \text{for } |x| < 1$$

- Ratio Test:
  For the series $\sum a_n$, suppose,

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L.$$

  - If $L < 1$, then the series converges.
  - If $L > 1$, then the series diverges.
  - If $L = 1$, then the test fails.

- *Useful Integrals for Comparison:*

  $\int_1^\infty \frac{1}{x^p}\,dx$ converges for $p > 1$ and diverges for $p \le 1$.

  $\int_0^1 \frac{1}{x^p}\,dx$ converges for $p < 1$ and diverges for $p \ge 1$.

  $\int_0^\infty e^{-ax}\,dx$ converges for $a > 0$.

- nth degree Taylor Polynomial of $f(x)$ centered at $x = a$:

  $f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$

- Taylor series of $f(x)$ centered at $x = a$:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

- Taylor Series of important functions centered at $x = 0$:

  $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

  $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

  $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

  $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$ for $-1 < x < 1$

  $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ for $-1 < x \le 1$

  $(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots$ for $-1 < x < 1$

- **Differentiation formulas**

$\frac{d}{dx}(x^n) = nx^{n-1}$ $\frac{d}{dx}(\ln\lvert x\rvert) = \frac{1}{x}$	$\frac{d}{dx}(e^x) = e^x$ $\frac{d}{dx}(\sin(x)) = \cos x$ $\frac{d}{dx}(\tan(x)) = \sec^2 x$ $\frac{d}{dx}(\sec(x)) = \sec x \tan x$	$\frac{d}{dx}(a^x) = (\ln a)a^x$ $\frac{d}{dx}(\cos(x)) = -\sin x$ $\frac{d}{dx}(\cot(x)) = -\csc^2 x$ $\frac{d}{dx}(\csc(x)) = -\csc x \cot x$
$\frac{d}{dx}(\arcsin(x)) = \frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}(\arccos(x)) = \frac{-1}{\sqrt{1-x^2}}$	$\frac{d}{dx}(\arctan(x)) = \frac{1}{1+x^2}$

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Here a, b, c, d are constants.

A Short Table of Indefinite Integrals

I. Basic Functions

1. $\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C, \ (n \neq -1)$
2. $\int \frac{1}{x} \, dx = \ln|x| + C$
3. $\int a^x \, dx = \frac{1}{\ln a} a^x + C$
4. $\int \ln x \, dx = x \ln x - x + C$
5. $\int \sin ax \, dx = -\frac{1}{a} \cos ax + C$
6. $\int \cos ax \, dx = \frac{1}{a} \sin ax + C$
7. $\int \tan ax \, dx = -\frac{1}{a} \ln|\cos ax| + C$

II. Products of e^x, $\cos x$, and $\sin x$

8. $\int e^{ax} \sin(bx) \, dx = \frac{1}{a^2+b^2} e^{ax}[a \sin(bx) - b \cos(bx)] + C$
9. $\int e^{ax} \cos(bx) \, dx = \frac{1}{a^2+b^2} e^{ax}[a \cos(bx) + b \sin(bx)] + C$
10. $\int \sin(ax) \sin(bx) \, dx = \frac{1}{b^2-a^2}[a \cos(ax) \sin(bx) - b \sin(ax) \cos(bx)] + C, \quad a \neq b$
11. $\int \cos(ax) \cos(bx) \, dx = \frac{1}{b^2-a^2}[b \cos(ax) \sin(bx) - a \sin(ax) \cos(bx)] + C, \quad a \neq b$
12. $\int \sin(ax) \cos(bx) \, dx = \frac{1}{b^2-a^2}[b \sin(ax) \sin(bx) + a \cos(ax) \cos(bx)] + C, \quad a \neq b$

III. Product of Polynomial $p(x)$ with $\ln x$, e^x, $\cos x$, and $\sin x$

13. $\int x^n \ln x \, dx = \frac{1}{n+1} x^{n+1} \ln x - \frac{1}{(n+1)^2} x^{n+1} + C, \quad n \neq -1, x > 0$
14. $\int p(x) e^{ax} \, dx = \frac{1}{a} p(x) e^{ax} - \frac{1}{a^2} p'(x) e^{ax} + \frac{1}{a^3} p''(x) e^{ax} - \cdots + C$

    $(+ - + - + - + \ldots)$ (signs alternate)
15. $\int p(x) \sin ax \, dx = -\frac{1}{a} p(x) \cos(ax) + \frac{1}{a^2} p'(x) \sin(ax) + \frac{1}{a^3} p''(x) \cos(ax) - \cdots + C$

    $(- + + - - + + - - \ldots)$ (signs alternate in pairs)
16. $\int p(x) \cos ax \, dx = \frac{1}{a} p(x) \sin(ax) + \frac{1}{a^2} p'(x) \cos(ax) - \frac{1}{a^3} p''(x) \sin(ax) - \cdots + C$

    $(+ + - - + + - - \ldots)$ (signs alternate in pairs)

IV. Integer Powers of $\sin x$ and $\cos x$

17. $\displaystyle\int \sin^n x\, dx = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x\, dx, \quad n \text{ positive}$
18. $\displaystyle\int \cos^n x\, dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x\, dx, \quad n \text{ positive}$
19. $\displaystyle\int \frac{1}{\sin^m x}\, dx = -\frac{1}{m-1}\frac{\cos x}{\sin^{m-1} x} + \frac{m-2}{m-1}\int \frac{1}{\sin^{m-2} x}\, dx, \quad m \neq 1, m \text{ positive}$
20. $\displaystyle\int \frac{1}{\sin x}\, dx = \frac{1}{2}\ln\left|\frac{\cos x - 1}{\cos x + 1}\right| + C$
21. $\displaystyle\int \frac{1}{\cos^m x}\, dx = \frac{1}{m-1}\frac{\sin x}{\cos^{m-1} x} + \frac{m-2}{m-1}\int \frac{1}{\cos^{m-2} x}\, dx, \quad m \neq 1, m \text{ positive}$
22. $\displaystyle\int \frac{1}{\cos x}\, dx = \frac{1}{2}\ln\left|\frac{\sin x + 1}{\sin x - 1}\right| + C$
23. $\displaystyle\int \sin^m x \cos^n x\, dx :$

    If n is odd, let $w = \sin x$.
    If both m and n are even and non-negative, convert all to $\sin x$ or all to $\cos x$ (using $\sin^2 x + \cos^2 x = 1$), and use IV-17 or IV-18.
    If m and n are even and one of them is negative, convert to whichever function is in the denominator and use IV-19 or IV-21.
    The case in which both m and n are even and negative is omitted.

V. Quadratic in the Denominator

24. $\displaystyle\int \frac{1}{x^2 + a^2}\, dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C,\ a \neq 0$
25. $\displaystyle\int \frac{bx + c}{x^2 + a^2}\, dx = \frac{b}{2}\ln|x^2 + a^2| + \frac{c}{a}\arctan\left(\frac{x}{a}\right) + C,\ a \neq 0$
26. $\displaystyle\int \frac{1}{(x-a)(x-b)}\, dx = \frac{1}{(a-b)}(\ln|x-a| - \ln|x-b|) + C, \quad a \neq b$
27. $\displaystyle\int \frac{cx + d}{(x-a)(x-b)}\, dx = \frac{1}{(a-b)}[(ac+d)\ln|x-a| - (bc+d)\ln|x-b|] + C, \quad a \neq b$

VI. Integrands involving $\sqrt{a^2+x^2}, \sqrt{a^2-x^2}, \sqrt{x^2-a^2}, a > 0$

28. $\displaystyle\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\left(\frac{x}{a}\right) + C$
29. $\displaystyle\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln|x + \sqrt{x^2 \pm a^2}| + C$
30. $\displaystyle\int \sqrt{a^2 \pm x^2}\, dx = \frac{1}{2}\left(x\sqrt{a^2 \pm x^2} + a^2\int \frac{1}{\sqrt{a^2 \pm x^2}}\, dx\right) + C$
31. $\displaystyle\int \sqrt{x^2 - a^2}\, dx = \frac{1}{2}\left(x\sqrt{x^2 - a^2} - a^2\int \frac{1}{\sqrt{x^2 - a^2}}\, dx\right) + C$