Polytechnic University

MA 1122 | FINAL EXAM | JULY, 2006

Print Name:

Signature:

ID #:

Instructor/Section:

Directions: You have **90 minutes** to answer the following questions. ***You must show all your work*** as neatly and clearly as possible and indicate the final answer clearly. You may use only a TI-30 calculator. The last pages contain formulas that you might find useful. You may tear those pages out.

If you are feeling ill you should inform the proctor. The proctor will note your name, Poly ID and accept any written statement(s) that you may wish to make regarding your illness.

Problem	Possible	Points
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total	100	

(1) (Page 355, Problems 5–32) Determine if each of the following integrals is convergent or divergent. If it is convergent, find its **exact** value. Show all your work.

(a) $\displaystyle\int_{-5}^{20} \frac{1}{\sqrt{x+5}}\,dx.$

(b) $\displaystyle\int_{2}^{\infty} \frac{x}{\sqrt{x^2-4}}\,dx.$

(2) (Page 380, Problems 11 and 12) Find the **exact** arc length of the function

$$f(x) = 5\sqrt{x^3}$$

from $x = 0$ to $x = 1$. Show all your work.

(3) (Page 373 , Problem 8; Page 380, Problem 30)

(a) Set up but do not evaluate the area bounded by the graphs of the equations $y = \sqrt{x}$, $y + x = 6$ and $y = 0$. You do not have to show your work.

(b) Set up but do not evaluate the volume of the solid generated by the region R bounded by $y = 3 + 2x - x^2$, $x = 0$, $y = 0$ and rotated around the line $y = -1$. You do not have to show your work.

(4) (Page 355 , Problems 5–32; Page 448, Problems 20 and 21, and Page 442, Problems 20–31) Fill in the blanks. You do not have to show your work.

(a) $\displaystyle\int_{1/2}^{2} \frac{1}{x(\ln x)^{1/5}}\,dx$ is ____________________(convergent/divergent).

(b) $\displaystyle\int_{1}^{\infty} \frac{\ln x}{e^{2x}}\,dx$ is ____________________(convergent/divergent).

(c) $\displaystyle\sum_{k=1}^{\infty} \left(\frac{e}{\pi}\right)^{k+1}$ is ____________________(convergent/divergent).

(d) $\displaystyle\sum_{k=1}^{\infty} \frac{2^k + 3^k}{4^k}$ is ____________________(convergent/divergent).

(e) The sequence $\displaystyle\left\{\frac{1-n^2}{2+3n^2}\right\}_{n=1}^{\infty}$ is ____________________(convergent/divergent).

(5) (Concepts) Determine whether each of the following statements is TRUE or FALSE. You do not have to explain.

(a) If f' is continuous on $[0, \infty)$ and $\lim_{x\to\infty} f(x) = 0$, then $\int_0^\infty f'(x)\,dx$ converges.

(b) If f is continuous on $[0, \infty)$ and $\lim_{x\to\infty} f(x) = 0$, then $\int_0^\infty f(x)\,dx$ converges.

(c) If f is an even function and $\int_0^\infty f(x)\,dx$ converges, then $\int_{-\infty}^\infty f(x)\,dx$ converges.

(d) If $\int_0^\infty f(x)\,dx$ is convergent, then $\int_0^\infty (\pi + f(x))\,dx$ is also convergent.

(e) If $0 \le a_n \le b_n$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

(6) (Page 448, Problems 11–21) Find the sum in each case if possible. If it is not possible, write "not possible". Show all your work.

(a) $-2\left(\frac{1}{10}\right)^5 - 2\left(\frac{1}{10}\right)^6 - 2\left(\frac{1}{10}\right)^7 \cdots - 2\left(\frac{1}{10}\right)^{20}$.

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{\sin^n\left(\frac{\pi}{6}\right) + \pi^n}{e^{2n}}$.

(7) (Practice Exam) Consider a population $P(t)$ with constant relative birth and death rates α and β, respectively, and a constant emigration rate m, where α, β and m are positive constants. Assume that $\alpha > \beta$. Then the rate of change of the population at time t is modeled by the differential equation

$$\frac{dP}{dt} = kP - m \qquad \text{where } k = \alpha - \beta.$$

Choose from the list below ***all*** solutions of the above equation that satisfy the initial condition $P(0) = P_0$.

You don't have to show work.

(a) $\left(e^{kt} - \frac{m}{P_0}\right) P_0 + m$

(b) $\left(P_0 - \frac{m}{k}\right) e^{kt} + \frac{m}{k}$

(c) $(P_0 - m)\, e^{kt} + m$

(d) $\frac{m}{k}\left(1 - e^{kt}\right) + P_0 e^{kt}$

(e) $(P_0 - k)\, e^{mt} + k$

(8) (Page 446, Problem 36) Find the **exact** value(s) of x in the interval $[0, \pi]$ by first identifying the left-hand-side in terms of Taylor series of a "known" function.

$$\frac{3^2}{2!}x^2 - \frac{3^4}{4!}x^4 + \frac{3^6}{6!}x^6 - \frac{3^8}{8!}x^8 + \cdots = \frac{1}{2}$$

You must show all your work.

(9) (Page 473, Problems 1–24) Determine whether each of the statements below is TRUE or FALSE. No explanation required.

(a) The Taylor polynomial for $\sin(x)$ about $x = \pi$ is

$$-(x-\pi) + \frac{(x-\pi)^3}{3!} - \frac{(x-\pi)^5}{5!} - \cdots.$$

(b) The Taylor polynomial of degree 2 for e^x near $x = -5$ is

$$1 + (x+5) + \frac{(x+5)^2}{2}.$$

(c) The Taylor series for $x^3 \cos(x)$ about $x = 0$ has nonzero coefficients only for odd powers of x.

(d) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = \frac{1}{e}.$

(e) If the Taylor series of $f(x)$ is $2x + 4x^2 + \frac{1}{5}x^3 - \cdots$, then $f'''(0) = \frac{1}{5}$.

(10) The table below gives some values of a function f and its derivatives.

x	$f(x)$	$f'(x)$	$f''(x)$	$f'''(x)$
0	2	1	4	7
1	0	-1.4	-3	-8
2	-2.4	4.8	-9	-2.1
3	-3.5	6.5	-3.7	-2

Suppose $f(x) = \int_{3x}^{3} h\left(\frac{t}{3}\right)\,dt$. Write the second degree Taylor polynomial for h about $x = 2$. You must show all your work.

Useful formulas

- *Physics formulas:*
  The *acceleration* due to gravity, g: $g = 9.8\text{m/sec}^2$, or $g = 32\text{ft/sec}^2$.

  Mass density of water $= 1000\ \text{kg/m}^3$, Weight density of water $= 62.4\ \text{lbs/ft}^3$.

  Force $=$ mass $\times$ acceleration Work $=$ Force $\times$ distance.

  The center of mass, $\bar{x}$, of an object lying on the x-axis between $x = a$ and $x = b$, with mass density $\delta(x)$ is given by $\bar{x} = \dfrac{\int_a^b x\delta(x)\,dx}{\text{total mass}}$.

  Arc length of a curve $y = f(x)$ from $x = a$ to $x = b$: $L = \displaystyle\int_a^b \sqrt{1 + (f'(x))^2}\,dx$

- *Integration by Parts*:
$$\int udv = uv - \int vdu \qquad \text{or} \qquad \int uv'dx = uv - \int vu'dx$$

- Finite Geometric Sum:
$$a + ax + ax^2 + \cdots + ax^{n-1} = \frac{a(1 - x^n)}{1 - x}$$

- Infinite Geometric Series:
$$a + ax + ax^2 + \cdots = \frac{a}{1 - x} \qquad \text{for } |x| < 1$$

- *Useful Integrals for Comparison:*

  $\int_1^\infty \frac{1}{x^p}\,dx$ converges for $p > 1$ and diverges for $p \le 1$.

  $\int_0^1 \frac{1}{x^p}\,dx$ converges for $p < 1$ and diverges for $p \ge 1$.

  $\int_0^\infty e^{-ax}\,dx$ converges for $a > 0$.

- **Differentiation formulas**

$\frac{d}{dx}(x^n) = nx^{n-1}$ $\frac{d}{dx}(\ln\|x\|) = \frac{1}{x}$	$\frac{d}{dx}(e^x) = e^x$ $\frac{d}{dx}(\sin(x)) = \cos x$	$\frac{d}{dx}(a^x) = (\ln a)a^x$ $\frac{d}{dx}(\cos(x)) = -\sin x$
	$\frac{d}{dx}(\tan(x)) = \sec^2 x$	$\frac{d}{dx}(\cot(x)) = -\csc^2 x$
	$\frac{d}{dx}(\sec(x)) = \sec x \tan x$	$\frac{d}{dx}(\csc(x)) = -\csc x \cot x$
$\frac{d}{dx}(\arcsin(x)) = \frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}(\arccos(x)) = \frac{-1}{\sqrt{1-x^2}}$	$\frac{d}{dx}(\arctan(x)) = \frac{1}{1+x^2}$

YOUR SIGNATURE: ______________________________

Here a, b, c, d are constants.

A Short Table of Indefinite Integrals

I. Basic Functions

1. $\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C, \ (n \neq -1)$
2. $\int \frac{1}{x} \, dx = \ln|x| + C$
3. $\int a^x \, dx = \frac{1}{\ln a} a^x + C$
4. $\int \ln x \, dx = x \ln x - x + C$
5. $\int \sin ax \, dx = -\frac{1}{a} \cos ax + C$
6. $\int \cos ax \, dx = \frac{1}{a} \sin ax + C$
7. $\int \tan ax \, dx = -\frac{1}{a} \ln|\cos ax| + C$

II. Products of e^x, $\cos x$, and $\sin x$

8. $\int e^{ax} \sin(bx) \, dx = \frac{1}{a^2+b^2} e^{ax} [a \sin(bx) - b \cos(bx)] + C$
9. $\int e^{ax} \cos(bx) \, dx = \frac{1}{a^2+b^2} e^{ax} [a \cos(bx) + b \sin(bx)] + C$
10. $\int \sin(ax) \sin(bx) \, dx = \frac{1}{b^2-a^2} [a \cos(ax) \sin(bx) - b \sin(ax) \cos(bx)] + C, \quad a \neq b$
11. $\int \cos(ax) \cos(bx) \, dx = \frac{1}{b^2-a^2} [b \cos(ax) \sin(bx) - a \sin(ax) \cos(bx)] + C, \quad a \neq b$
12. $\int \sin(ax) \cos(bx) \, dx = \frac{1}{b^2-a^2} [b \sin(ax) \sin(bx) + a \cos(ax) \cos(bx)] + C, \quad a \neq b$

III. Product of Polynomial $p(x)$ with $\ln x$, e^x, $\cos x$, and $\sin x$

13. $\int x^n \ln x \, dx = \frac{1}{n+1} x^{n+1} \ln x - \frac{1}{(n+1)^2} x^{n+1} + C, \quad n \neq -1, x > 0$
14. $\int p(x) e^{ax} \, dx = \frac{1}{a} p(x) e^{ax} - \frac{1}{a^2} p'(x) e^{ax} + \frac{1}{a^3} p''(x) e^{ax} - \cdots + C$

    $(+ - + - + - + \ldots)$ (signs alternate)
15. $\int p(x) \sin ax \, dx = -\frac{1}{a} p(x) \cos(ax) + \frac{1}{a^2} p'(x) \sin(ax) + \frac{1}{a^3} p''(x) \cos(ax) - \cdots + C$

    $(- + + - - + + - - \ldots)$ (signs alternate in pairs)
16. $\int p(x) \cos ax \, dx = \frac{1}{a} p(x) \sin(ax) + \frac{1}{a^2} p'(x) \cos(ax) - \frac{1}{a^3} p''(x) \sin(ax) - \cdots + C$

    $(+ + - - + + - - \ldots)$ (signs alternate in pairs)

IV. Integer Powers of $\sin x$ and $\cos x$

17. $\displaystyle\int \sin^n x\ dx = -\frac{1}{n}\sin^{n-1} x\cos x + \frac{n-1}{n}\int \sin^{n-2} x\ dx,\quad n \text{ positive}$
18. $\displaystyle\int \cos^n x\ dx = \frac{1}{n}\cos^{n-1} x\sin x + \frac{n-1}{n}\int \cos^{n-2} x\ dx,\quad n \text{ positive}$
19. $\displaystyle\int \frac{1}{\sin^m x}\ dx = -\frac{1}{m-1}\frac{\cos x}{\sin^{m-1} x} + \frac{m-2}{m-1}\int \frac{1}{\sin^{m-2} x}\ dx,\quad m \neq 1, m \text{ positive}$
20. $\displaystyle\int \frac{1}{\sin x}\ dx = \frac{1}{2}\ln\left|\frac{\cos x - 1}{\cos x + 1}\right| + C$
21. $\displaystyle\int \frac{1}{\cos^m x}\ dx = \frac{1}{m-1}\frac{\sin x}{\cos^{m-1} x} + \frac{m-2}{m-1}\int \frac{1}{\cos^{m-2} x}\ dx,\quad m \neq 1, m \text{ positive}$
22. $\displaystyle\int \frac{1}{\cos x}\ dx = \frac{1}{2}\ln\left|\frac{\sin x + 1}{\sin x - 1}\right| + C$
23. $\displaystyle\int \sin^m x\cos^n x\ dx :$

    If n is odd, let $w = \sin x$.
    If both m and n are even and non-negative, convert all to $\sin x$ or all to $\cos x$ (using $\sin^2 x + \cos^2 x = 1$), and use IV-17 or IV-18.
    If m and n are even and one of them is negative, convert to whichever function is in the denominator and use IV-19 or IV-21.
    The case in which both m and n are even and negative is omitted.

V. Quadratic in the Denominator

24. $\displaystyle\int \frac{1}{x^2 + a^2}\ dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C,\ a \neq 0$
25. $\displaystyle\int \frac{bx + c}{x^2 + a^2}\ dx = \frac{b}{2}\ln|x^2 + a^2| + \frac{c}{a}\arctan\left(\frac{x}{a}\right) + C,\ a \neq 0$
26. $\displaystyle\int \frac{1}{(x-a)(x-b)}\ dx = \frac{1}{(a-b)}(\ln|x-a| - \ln|x-b|) + C,\quad a \neq b$
27. $\displaystyle\int \frac{cx + d}{(x-a)(x-b)}\ dx = \frac{1}{(a-b)}[(ac+d)\ln|x-a| - (bc+d)\ln|x-b|] + C,\quad a \neq b$

VI. Integrands involving $\sqrt{a^2+x^2}, \sqrt{a^2-x^2}, \sqrt{x^2-a^2}, a > 0$

28. $\displaystyle\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\left(\frac{x}{a}\right) + C$
29. $\displaystyle\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln|x + \sqrt{x^2 \pm a^2}| + C$
30. $\displaystyle\int \sqrt{a^2 \pm x^2}\ dx = \frac{1}{2}\left(x\sqrt{a^2 \pm x^2} + a^2\int \frac{1}{\sqrt{a^2 \pm x^2}}\ dx\right) + C$
31. $\displaystyle\int \sqrt{x^2 - a^2}\ dx = \frac{1}{2}\left(x\sqrt{x^2 - a^2} - a^2\int \frac{1}{\sqrt{x^2 - a^2}}\ dx\right) + C$