Chapter 3
Determinant

3.1 The Determinant Function

We follow an intuitive approach to introduce the definition of determinant. We already have a function defined on certain matrices: the trace. The trace assigns a number to a square matrix by summing the entries along the main diagonal of the matrix. So the trace is a function; its domain is the set of all square matrices, its range is the set of numbers. We also showed that the trace is a linear function, that is

\[
\text{tr}(A + B) = \text{tr}(A) + \text{tr}(B) \\
\text{tr}(cA) = c\text{tr}(A)
\]

where \(A\) and \(B\) are two \(n \times n\) matrices, and \(c\) is a constant.

The determinant is also a function which assigns a number to every square matrix. So its domain will be again the set of square matrices, and its range is the set of numbers. The notation of the determinant of a matrix \(A\) is

\[
\det(A).
\]

The determinant function has some nice properties, and we should emphasize two of them at this point:

\[
\det(AB) = \det(A) \det(B) \\
\det(I) = 1
\]

where \(A\) and \(B\) are two \(n \times n\) matrices. We will see the other properties later in this chapter.
### 3.2 Calculating the Determinant for $2 \times 2$ Matrices

**Definition 3.2.1.** The determinant of a $2 \times 2$ matrix is defined as

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$ 

**Example 3.2.1.**

$$\det(I_2) = \det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1.$$ 

**Example 3.2.2.**

$$\det \begin{pmatrix} 3 & 2 \\ -2 & 4 \end{pmatrix} = 3 \cdot 4 - 2 \cdot (-2) = 16.$$ 

**Theorem 3.2.1.** For any two $2 \times 2$ matrices $A$ and $B$

$$\det(AB) = \det(A)\det(B).$$ 

**Proof.** Let

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}. $$

Then

$$\det(A)\det(B) = (a_{11}a_{22} - a_{12}a_{21})(b_{11}b_{22} - b_{12}b_{21})$$

$$= a_{11}a_{22}b_{11}b_{22} - a_{11}a_{22}b_{12}b_{21} - a_{12}a_{21}b_{11}b_{22} + a_{12}a_{21}b_{12}b_{21}. \quad (3.2.1)$$

The product of the two matrices is

$$AB = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix},$$

so

$$\det(AB) = (a_{11}b_{11} + a_{12}b_{21})(a_{21}b_{12} + a_{22}b_{22}) - (a_{11}b_{12} + a_{12}b_{22})(a_{21}b_{11} + a_{22}b_{21})$$

$$= a_{11}b_{11}a_{21}b_{12} + a_{11}b_{11}a_{22}b_{22} + a_{12}b_{21}a_{21}b_{12} + a_{12}b_{21}a_{22}b_{22}$$

$$- a_{11}b_{12}a_{21}b_{11} - a_{11}b_{12}a_{22}b_{21} - a_{12}b_{22}a_{21}b_{11} - a_{12}b_{22}a_{22}b_{21}$$

$$= a_{11}b_{11}a_{22}b_{22} + a_{12}b_{21}a_{21}b_{12} - a_{11}b_{12}a_{22}b_{21} - a_{12}b_{22}a_{21}b_{11}. \quad (3.2.2)$$

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As we see the terms in 3.2.1 and in 3.2.2 are the same, so
\[
\det(A) \det(B) = \det(AB).
\]

If \( A \) is an invertible matrix, then we can apply this theorem for \( A \) and \( A^{-1} \):
\[
\det(AA^{-1}) = \det(A) \det(A^{-1}).
\]
The left hand side of the equation is \( \det(AA^{-1}) = \det(I) = 1 \), so we have that
\[
1 = \det(A) \det(A^{-1}).
\]
From this equation we can conclude that if \( A \) is invertible, then \( \det(A) \neq 0 \), \( \det(A^{-1}) \neq 0 \), and
\[
\det(A^{-1}) = \frac{1}{\det(A)}.
\]

**Corollary 3.2.2.** If \( A \) is invertible, then
\[
\det(A^{-1}) = \frac{1}{\det(A)}.
\]

Using the definition of the determinant we can easily show some properties of the determinant function for \( 2 \times 2 \) matrices.

**Corollary 3.2.3.**
\[
\det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1.
\]

**Corollary 3.2.4.** If a row or column of a matrix is 0, then the determinant of the matrix is 0.

**Corollary 3.2.5.** If a matrix is triangular, then its determinant is the product of the diagonal entries:
\[
\det \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} = ad,
\]
\[
\det \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = ad.
\]
Corollary 3.2.6. If we swap two rows or two columns, then the determinant changes its sign:

\[
\det \begin{pmatrix} c & d \\ a & b \end{pmatrix} = - \det \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \\
\det \begin{pmatrix} b & d \\ a & c \end{pmatrix} = - \det \begin{pmatrix} a & b \\ c & d \end{pmatrix}.
\]

Corollary 3.2.7. If we multiply a row or column by a number \( k \), then the determinant will be \( k \) times as before:

\[
\det \begin{pmatrix} ka & kb \\ c & d \end{pmatrix} = k \det \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \\
\det \begin{pmatrix} ka & b \\ kc & d \end{pmatrix} = k \det \begin{pmatrix} a & b \\ c & d \end{pmatrix}.
\]

Corollary 3.2.8. If \( A \) is a \( 2 \times 2 \) matrix and \( c \) is a scalar, then \( \det(cA) = c^2 \det(A) \).

Corollary 3.2.9. If we add \( k \) times a row (or column) to the other row (or column), then the determinant will no change:

\[
\det \begin{pmatrix} a + kc & b + kd \\ c & d \end{pmatrix} = (a + kc)d - (b + kd)c \\
= ad - bc \\
= \det \begin{pmatrix} a & b \\ c & d \end{pmatrix}.
\]

3.3 Geometric Meaning of the Determinant

We can consider a \( 2 \times 2 \) matrix

\[
\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}
\]

as a collection of two column vectors

\[
\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{and} \quad \vec{y} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.
\]
We can draw these two vectors in the Descartes coordinate system, and we can see that any two such vectors determine a parallelogram. (You may see Section 4.1 for more about vectors.)

**Example 3.3.1.** The area of the parallelogram with vertices $(0,0)$, $(1,0)$, $(1,0.8)$ and $(2,0.8)$ is $0.8$, see Figure 3.1. This is the same as the determinant of the matrix formed by the two column vectors which determine the parallelogram:

$$\det \begin{pmatrix} 1 & 1 \\ 0 & 0.8 \end{pmatrix} = 0.8.$$ 

**Example 3.3.2.** The area of the parallelogram with vertices $(1,1)$, $(2,1)$, $(2,1.8)$ and $(3,1.8)$ is the same as in the previous example, since this parallelogram is just shifted, but its area has not changed.

**Theorem 3.3.1.** *The determinant of a $2 \times 2$ matrix $\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}$ is the signed area of the parallelogram determined by the two column vectors of the matrix. The sign is positive if the angle that you get by rotating the first column vector toward the second column vector in the counterclockwise direction is less than $\pi$. The sign is negative if this angle is greater than $\pi$.***

*Proof.* We will only give a visual verification of this theorem here by cutting and moving pieces of the parallelogram around, see Figure 3.2. On each picture, the dark shaded part has the same area as the lightly shaded piece. \qed
Figure 3.2: Verifying Theorem 3.3.1
3.4 Properties of the Determinant Function

We defined the determinant function for $2 \times 2$ matrices, and saw some nice properties of it. We would like to extend our definition for all square matrices, so that these properties remain true.

**Theorem 3.4.1.** The determinant assigns a number for every square matrix, with the following properties. Let $A$ and $B$ be two $n \times n$ matrices.

1. $\det(I_n) = 1$.

2. If $B$ is the matrix that results when a single row or single column of $A$ is multiplied by a scalar $k$, then $\det(B) = k \det(A)$.

3. If $B$ is the matrix that results when two rows or two columns of $A$ are interchanged, then $\det(B) = -\det(A)$.

4. If $B$ is the matrix that results when a multiple of one row is added to another row or when a multiple of one column of $A$ is added to another column, then $\det(B) = \det(A)$.

Some further properties of the determinant that we showed for $2 \times 2$ matrices and remain true for larger matrices:

**Theorem 3.4.2.** Let $A$ and $B$ be $n \times n$ matrices.

1. If $A$ has a row or a column of zeroes, then $\det(A) = 0$.

2. If $A$ is a triangular matrix, then its determinant is the product of the diagonal entries.

3. $\det(A) = \det(A^T)$.

4. $\det(AB) = \det(A) \det(B)$.

5. A square matrix $A$ is invertible if and only if $\det(A) \neq 0$.

6. If $A$ is invertible, then $\det(A^{-1}) = \frac{1}{\det(A)}$.

7. If $k$ is a scalar, then $\det(kA) = k^n \det(A)$.
3.5 Evaluating the Determinant by Row Reduction

Theorem 3.4.1 allows us to use row- (or column-) reduction to calculate the value of the determinant of larger matrices than $2 \times 2$. The goal is to reduce the matrix to a triangular form, because we know that the determinant of a triangular matrix is the product of its entries along the main diagonal.

Example 3.5.1. Using row reduction let’s calculate the determinant of the matrix

$$
\begin{pmatrix}
1 & 1 & 1 & 1 \\
x & 1 & 1 & 1 \\
x^2 & x^2 & 1 & 1 \\
x^3 & x^3 & x^3 & 1
\end{pmatrix}
$$

If we do the following row operations $R_2 = R_2 - xR_1$, $R_3 = R_3 - x^2R_1$, $R_4 = R_4 - x^3R_1$, then the value of the determinant does not change.

$$
\text{det}
\begin{pmatrix}
1 & 1 & 1 & 1 \\
x & 1 & 1 & 1 \\
x^2 & x^2 & 1 & 1 \\
x^3 & x^3 & x^3 & 1
\end{pmatrix}
= \text{det}
\begin{pmatrix}
1 & 1 & 1 & 1 \\
0 & 1 - x & 1 & 1 \\
0 & 0 & 1 - x^2 & 1 \\
0 & 0 & 0 & 1 - x^3
\end{pmatrix}
= (1 - x)(1 - x^2)(1 - x^3).
$$

Example 3.5.2. Since we already calculated the determinant of

$$
A =
\begin{pmatrix}
1 & 1 & 1 & 1 \\
x & 1 & 1 & 1 \\
x^2 & x^2 & 1 & 1 \\
x^3 & x^3 & x^3 & 1
\end{pmatrix},
$$

we can easily tell that this matrix is singular (has no inverse) over the real numbers if the determinant is equal to zero, that is if $x = \pm 1$.

Notice, that $\text{det}(A) = 0$ has two more complex roots, so over the complex numbers $A$ is singular not only for $x = \pm 1$ but also for $x = e^{i\frac{2\pi}{3}}$, or $x = e^{i\frac{4\pi}{3}}$. 
3.6 Determinant, Invertibility and Systems of Linear Equations

Every system of linear equations can be written in a matrix form:

\[ A\vec{x} = \vec{b}, \]

where \( A \) is the coefficient matrix, \( \vec{x} \) is the vector of unknowns, and \( \vec{b} \) is the vector of constant terms. You can always solve a system of linear equations using the Gaussian algorithm, by reducing it to row echelon form or to the reduced row echelon form. However if the coefficient matrix \( A \) is a square matrix, then calculating the determinant might be useful.

**Theorem 3.6.1.** If \( A \) is an \( n \times n \) matrix, then the following are equivalent. (That is, if one of these statements is true, then all the others must also be true.)

1. \( A \) is invertible.
2. \( \det(A) \neq 0. \)
3. \( A\vec{x} = \vec{b} \) has exactly one solution for every \( n \times 1 \) matrix \( \vec{b} \).
4. \( A\vec{x} = \vec{0} \) has only the trivial solution (that is the solution is the zero vector).
5. The reduced row-echelon form of \( A \) is \( I_n \).

**Proof.** The idea of the proof: if \( A \) is invertible, then we can multiply both sides of the equation

\[ A\vec{x} = \vec{b} \]

by \( A^{-1} \), and we get

\[ A^{-1}A\vec{x} = A^{-1}\vec{b}. \]

Since \( A^{-1}A = I \), we solved the equation:

\[ \vec{x} = A^{-1}\vec{b}, \]

and there is only one solution for every \( \vec{b} \). If \( \vec{b} \) was the zero vector, then the solution is

\[ \vec{x} = A^{-1}\vec{0} = \vec{0}, \]

the trivial solution (each of its component is 0). \( \square \)
Corollary 3.6.2. If $A$ is an $n \times n$ matrix such that $\det(A) = 0$, then the equation

1. $A\vec{x} = \vec{0}$ has a non-trivial solution (that is a solution whose components are not all 0).

2. $A\vec{x} = \vec{b}$ has either more than one solution for a non-zero $n \times 1$ matrix $\vec{b}$, or has no solutions at all. (We have to use the Gaussian algorithm to find out what is happening in this case.)

Example 3.6.1. The linear system

$$
\begin{align*}
2kx + (k + 1)y &= 2 \\
(k + 6)x + (k + 3)y &= 3
\end{align*}
$$

has exactly one solution if

$$
\det \begin{pmatrix} 2k & k + 1 \\ k + 6 & k + 3 \end{pmatrix} \neq 0,
$$

that is when $2k(k + 3) - (k + 1)(k + 6) = k^2 - k - 6 = (k - 3)(k + 2) \neq 0$, i.e. if $k \neq -2, 3$. However if $k = -2$ or 3, then we have to use the Gaussian algorithm.

If $k = -2$, then the system becomes

$$
\begin{align*}
-4x - y &= 2 \\
4x + y &= 3
\end{align*}
$$

whose row echelon form is

$$
\begin{pmatrix} 1 & 1/4 & -1/2 \\ 0 & 0 & 5 \end{pmatrix},
$$

and we can conclude that the system has no solution.

If $k = 3$, then the system becomes

$$
\begin{align*}
6x + 4y &= 2 \\
9x + 6y &= 3
\end{align*}
$$

whose row echelon form is

$$
\begin{pmatrix} 1 & 2/3 & 1/3 \\ 0 & 0 & 0 \end{pmatrix}.
$$

In this case $y$ is a free variable, and the system has infinitely many solutions over $\mathbb{R}$ and also over $\mathbb{C}$.
3.7 Cofactor Expansion, Adjoint Matrix

Definition 3.7.1. The recursive definition of the determinant using cofactor expansion along the $i$th row of $A$:

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + a_{i3}C_{i3} + \cdots a_{in}C_{in}.$$ 

With sum notation:

$$\det(A) = \sum_{k=1}^{n} (-1)^{i+k} a_{ik}C_{ik}.$$ 

The recursive definition of the determinant using cofactor expansion along the $j$th column of $A$:

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + a_{3j}C_{3j} + \cdots a_{nj}C_{nj}.$$ 

With sum notation:

$$\det(A) = \sum_{k=1}^{n} (-1)^{k+j} a_{kj}C_{kj}.$$ 

Here $C_{ij}$ denotes the cofactor of the entry $a_{ij}$: the determinant of the minor you get from $A$ by cancelling the $i$th row and $j$th column, with a plus or minus sign according to the “checkerboard”:

$$\begin{pmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

Definition 3.7.2. The cofactor matrix of $A$:

$$\begin{pmatrix} C_{11} & C_{12} & C_{13} & \cdots & C_{1n} \\ C_{21} & C_{22} & C_{23} & \cdots & C_{2n} \\ C_{31} & C_{32} & C_{33} & \cdots & C_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ C_{n1} & C_{n2} & C_{n3} & \cdots & C_{nn} \end{pmatrix}$$

is the matrix you get by replacing each entry in $A$ by its cofactor.
**Definition 3.7.3.** The *adjoint matrix* of $A$ is the transpose of the cofactor matrix:

$$
\text{adj}(A) = \begin{pmatrix}
C_{11} & C_{21} & C_{31} & \cdots & C_{n1} \\
C_{12} & C_{22} & C_{32} & \cdots & C_{n2} \\
C_{13} & C_{23} & C_{33} & \cdots & C_{n3} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
C_{1n} & C_{2n} & C_{3n} & \cdots & C_{nn}
\end{pmatrix}.
$$

**Theorem 3.7.1.**

$$
A \cdot \text{adj}(A) = \begin{pmatrix}
det(A) & 0 & 0 & \cdots & 0 \\
0 & det(A) & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & det(A)
\end{pmatrix} = det(A) \cdot I_n.
$$

**Theorem 3.7.2.** If $A$ is an invertible matrix, then

$$
A^{-1} = \frac{1}{\det(A)} \text{adj}(A).
$$

**Example 3.7.1.** Let

$$
A = \begin{pmatrix}
1 & 1 & 0 \\
3 & 4 & 5 \\
3 & 2 & 1
\end{pmatrix}.
$$

To get the determinant of $A$ we can use cofactor expansion. The first row would be the best choice, since it has a zero in it and the other entries are 1, which makes the calculations easier:

$$
\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}
= 1 \cdot \det \begin{pmatrix}
4 & 5 \\
2 & 1
\end{pmatrix} + 1 \cdot (-1) \det \begin{pmatrix}
3 & 5 \\
3 & 1
\end{pmatrix} + 0
= -6 + 12
= 6.
$$

The cofactor matrix of $A$ is:

$$
\begin{pmatrix}
-6 & 12 & -6 \\
-1 & 1 & 1 \\
5 & -5 & 1
\end{pmatrix}.
$$
The adjoint of $A$ is:

$$\text{adj}(A) = \begin{pmatrix} -6 & -1 & 5 \\ 12 & 1 & -5 \\ -6 & 1 & 1 \end{pmatrix}.$$  

Calculating $A \cdot \text{adj}(A)$:

$$A \cdot \text{adj}(A) = \begin{pmatrix} 1 & 1 & 0 \\ 3 & 4 & 5 \\ 3 & 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} -6 & -1 & 5 \\ 12 & 1 & -5 \\ -6 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix} = \det(A) \cdot I_3.$$  

Therefore we can use the adjoint matrix to find the inverse of $A$:

$$A^{-1} = \frac{1}{\det(A)} \cdot \text{adj}(A) = \frac{1}{6} \cdot \begin{pmatrix} -6 & -1 & 5 \\ 12 & 1 & -5 \\ -6 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -1/6 & 5/6 \\ 2 & 1/6 & -5/6 \\ -1 & 1/6 & 1/6 \end{pmatrix}.$$  

### 3.8 Calculating the Determinant for $3 \times 3$ Matrices

**Theorem 3.8.1 (The Sarrus’s rule).** To find the determinant of a $3 \times 3$ matrix $A$, write the first two column of $A$ to the right of $A$. Then, multiply the entries along as the diagram shows,

\[ a_{11} a_{22} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33}. \]

Please note, that Sarrus’s rule works **only for** $3 \times 3$ matrices. For larger matrices you either have to use row- or column reduction, cofactor expansion or see if the matrix is block triangular.
### 3.9 Block-Triangular Matrices

If $A$ is a block-triangular matrix, then the determinant of $A$ is the product of the determinants of its diagonal blocks.

**Example 3.9.1.**

\[
\begin{vmatrix}
1 & 1 & 0 & 4 & 5 & 6 \\
3 & 4 & 5 & 9 & 4 & 6 \\
3 & 2 & 1 & 4 & 3 & 1 \\
0 & 0 & 0 & -1 & 0 & -2 \\
0 & 0 & 0 & 0 & 2 & 1 \\
0 & 0 & 0 & 0 & 5 & 2 \\
\end{vmatrix}
= \det \begin{pmatrix}
1 & 1 & 0 \\
3 & 4 & 5 \\
3 & 2 & 1 \\
\end{pmatrix}
\det (-1)
\det \begin{pmatrix}
2 & 1 \\
5 & 2 \\
\end{pmatrix}
= 6 \cdot (-1) \cdot (-1)
= 6.
\]