

Chapter 5

Eigenvalues and Eigenvectors

5.1 Eigenvalues and Eigenvectors

Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a linear transformation. Then T can be represented by a matrix (the standard matrix), and we can write

$$T(\vec{v}) = A\vec{v}.$$

Example 5.1.1. Consider the transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ given by its standard matrix

$$A = \begin{pmatrix} 3 & -1 \\ 5 & -3 \end{pmatrix},$$

and let's calculate the image of some vectors under the transformation T .

$$\begin{aligned} T \begin{pmatrix} 1 \\ 1 \end{pmatrix} &= \begin{pmatrix} 3 & -1 \\ 5 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \\ T \begin{pmatrix} 3 \\ 3 \end{pmatrix} &= \begin{pmatrix} 3 & -1 \\ 5 & -3 \end{pmatrix} \begin{pmatrix} 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \end{pmatrix}. \end{aligned}$$

We may notice that the image of (x, x) is $2(x, x)$. Let's calculate some more images:

$$\begin{aligned} T \begin{pmatrix} 1 \\ 5 \end{pmatrix} &= \begin{pmatrix} 3 & -1 \\ 5 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ 5 \end{pmatrix} = \begin{pmatrix} -2 \\ -10 \end{pmatrix}, \\ T \begin{pmatrix} -2 \\ -10 \end{pmatrix} &= \begin{pmatrix} 3 & -1 \\ 5 & -3 \end{pmatrix} \begin{pmatrix} -2 \\ -10 \end{pmatrix} = \begin{pmatrix} 4 \\ 20 \end{pmatrix}. \end{aligned}$$

We may notice that the image of a vector $(x, 5x)$ is $-2(x, 5x)$. A couple of more

images:

$$T \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 5 & -3 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix},$$

$$T \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 5 & -3 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ -8 \end{pmatrix}.$$

There are no such nice patterns for these vectors.

Although a transformation given by a matrix A may move vectors in a variety direction, it often happens that there are special vectors on which the action is quite simple. In this section we would like to find those nonzero vectors \vec{v} , which are mapped to a scalar multiple of itself, that is

$$A\vec{v} = \lambda\vec{v}$$

for some scalar λ . In our example above, these vectors are (x, x) and $(x, 5x)$, where x can be any nonzero number.

Definition 5.1.1. If A is an $n \times n$ matrix, then a nonzero vector \vec{v} in \mathbb{R}^n is called an *eigenvector* of A if there is a scalar λ such that

$$A\vec{v} = \lambda\vec{v}.$$

The scalar λ is called an *eigenvalue* of A , and \vec{v} is said to be an eigenvector of A corresponding to λ .

We emphasize that eigenvectors are nonzero vectors. So the question is: when can we find a nonzero vector \vec{v} which satisfies the matrix equation $A\vec{v} = \lambda\vec{v}$ with some scalar λ ? Let's rearrange this equation $A\vec{v} = \lambda\vec{v}$ to

$$A\vec{v} - \lambda\vec{v} = \vec{0}.$$

Then, we can factor \vec{v} from both terms of the left hand side. However we have to be careful, because these products are not commutative, so we have to keep the order, and we will also have to write λI (a matrix) instead of λ , which is only a number. So we get

$$(A - \lambda I)\vec{v} = \vec{0}.$$

This is a homogeneous equation $B\vec{v} = \vec{0}$ with $B = A - \lambda I$. This homogeneous linear system has nonzero solutions, if $\det(B) = 0$. That is if $\det(A - \lambda I) = 0$.

So here is the idea: first we find those values of λ for which $\det(A - \lambda I) = 0$. Then for a such value of λ we solve the linear system $(A - \lambda I)\vec{v} = \vec{0}$ to get an eigenvector.

Definition 5.1.2. The equation $\det(A - \lambda I) = 0$ is called the *characteristic equation* of A . When expanded, the determinant $\det(A - \lambda I)$ is a polynomial in λ . This is called the *characteristic polynomial* of A .

Definition 5.1.3. The eigenvectors corresponding to λ are the nonzero vectors in the solution space of $(A - \lambda I)\vec{v} = \vec{0}$. We call this solution space the *eigenspace* of A corresponding to λ .

Remark 5.1.1. In some books you will find that the characteristic polynomial is defined by $\det(\lambda I - A)$. Using this as a definition, the characteristic polynomial would have 1 as its leading coefficient. You can show that the polynomials $\det(\lambda I - A)$ and $\det(A - \lambda I)$ differ only by a negative sign if the size of A is odd. If the size of A is even, then the two polynomials are the same.

5.2 Examples

Example 5.2.1. Let

$$A = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}$$

The characteristic polynomial of A is

$$\det(A - \lambda I) = \det \begin{pmatrix} -\lambda & 0 & -2 \\ 1 & 2 - \lambda & 1 \\ 1 & 0 & 3 - \lambda \end{pmatrix} = -\lambda^3 + 5\lambda^2 - 8\lambda + 4.$$

To get the eigenvalues, find the zeroes of the characteristic polynomial:

$$-\lambda^3 + 5\lambda^2 - 8\lambda + 4 = -(\lambda - 2)^2(\lambda - 1),$$

the eigenvalues of A are: $\lambda = 2$, which has *algebraic multiplicity* of 2, (that is $\lambda = 2$ is a double root of the characteristic equation) and $\lambda = 1$, which has algebraic multiplicity of 1 (that is $\lambda = 1$ is a simple root of the characteristic equation).

Let's find the eigenspace and a basis for the eigenspace for each of the eigenvalues of A . To find the eigenspace corresponding to λ , we have to find the solutions space of the equation $(A - \lambda I)\vec{v} = \vec{0}$. So for $\lambda = 2$, the augmented matrix is:

$$\left(\begin{array}{ccc|c} -2 & 0 & -2 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array} \right)$$

whose row-echelon form is

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right).$$

Since there are two free variables the solution space of $(A - 2I)\vec{v} = \vec{0}$, and therefore the eigenspace of A corresponding to $\lambda = 2$ has dimension two. The *geometric multiplicity* of the eigenvalue $\lambda = 2$ is 2 (the dimension of the corresponding eigenspace). The solutions of $(A - 2I)\vec{v} = \vec{0}$ are $(-v_3, v_2, v_3)$ where v_2 and v_3 are free variables. These are the eigenvectors of A corresponding to $\lambda = 2$. The eigenspace corresponding to $\lambda = 2$ is

$$\left\{ \begin{pmatrix} -v_3 \\ v_2 \\ v_3 \end{pmatrix} : v_2, v_3 \in \mathbb{C} \right\}.$$

A basis for the eigenspace is

$$\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

To find the eigenspace corresponding to $\lambda = 1$ we have to repeat the same procedure. We have to find the solutions space of the equation $(A - 1I)\vec{v} = \vec{0}$, the augmented matrix is:

$$\left(\begin{array}{ccc|c} -1 & 0 & -2 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \end{array} \right),$$

whose row-echelon form is

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right).$$

Since there is only one free variable the solution space of $(A - I)\vec{v} = \vec{0}$, and therefore the eigenspace of A corresponding to $\lambda = 1$ has dimension one. The geometric multiplicity of the eigenvalue $\lambda = 1$ is 1. The solutions of $(A - I)\vec{v} = \vec{0}$ are $(-2v_3, v_3, v_3)$, where v_3 is a free variable. These are the eigenvectors of A corresponding to $\lambda = 1$. The eigenspace corresponding to $\lambda = 1$ is

$$\left\{ \begin{pmatrix} -2v_3 \\ v_3 \\ v_3 \end{pmatrix} : v_3 \in \mathbb{C} \right\}.$$

A basis for the eigenspace is

$$\left\{ \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

Example 5.2.2. Let

$$B = \begin{pmatrix} 5 & 0 & 4 \\ 0 & 3 & -1 \\ 0 & 0 & -2 \end{pmatrix}.$$

It is a triangular matrix. The eigenvalues of B are $\lambda = 5, 3,$ and -2 . Each has algebraic multiplicity of one. For each eigenvalue we can find the eigenspace, and a basis for the eigenspace. The eigenspace corresponding to $\lambda = 5$ is

$$\left\{ \begin{pmatrix} v_1 \\ 0 \\ 0 \end{pmatrix} : v_1 \in \mathbb{C} \right\}.$$

A basis for the eigenspace is

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

The eigenspace corresponding to $\lambda = 3$ is

$$\left\{ \begin{pmatrix} 0 \\ v_2 \\ 0 \end{pmatrix} : v_2 \in \mathbb{C} \right\}.$$

A basis for the eigenspace is

$$\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

The eigenspace corresponding to $\lambda = -2$ is

$$\left\{ \begin{pmatrix} -\frac{4}{7}v_3 \\ \frac{1}{5}v_3 \\ v_3 \end{pmatrix} : v_3 \in \mathbb{C} \right\}.$$

A basis for the eigenspace is

$$\left\{ \begin{pmatrix} -\frac{4}{7} \\ \frac{1}{5} \\ 1 \end{pmatrix} \right\}$$

or a more convenient one is: $(-20, 7, 35)$.

Example 5.2.3. Let

$$C = \begin{pmatrix} 5 & -1 \\ 1 & 3 \end{pmatrix}.$$

The characteristic polynomial of C is

$$\det(C - \lambda I) = \det \begin{pmatrix} 5 - \lambda & -1 \\ 1 & 3 - \lambda \end{pmatrix} = \lambda^2 - 8\lambda + 16.$$

To find the eigenvalues we have to find the roots of the characteristic polynomial

$$\lambda^2 - 8\lambda + 16 = (\lambda - 4)^2,$$

so C has only one eigenvalue $\lambda = 4$, which has algebraic multiplicity of two (i.e. it is a double root of the characteristic equation).

To find the eigenspace corresponding to $\lambda = 4$ we have to find the solutions space of the equation $(4I - A)\vec{v} = \vec{0}$, the augmented matrix is:

$$\begin{pmatrix} -1 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix}$$

whose row-echelon form is

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since there is only one free variable the solution space, and therefore the eigenspace corresponding to $\lambda = 4$ has dimension one. The geometric multiplicity of the eigenvalue $\lambda = 4$ is 1. The solutions, so the eigenvectors are (v_2, v_2) , where v_2 is a free variable. The eigenspace corresponding to $\lambda = 4$ is

$$\left\{ \begin{pmatrix} v_2 \\ v_2 \end{pmatrix} : v_2 \in \mathbb{C} \right\}.$$

A basis for the eigenspace is

$$\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}.$$

Example 5.2.4. Let

$$N = \begin{pmatrix} 2 & 3 & -1 \\ 0 & -4 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

a triangular matrix. Then the matrix $N - \lambda I$

$$N - \lambda I = \begin{pmatrix} 2 - \lambda & 3 & -1 \\ 0 & -4 - \lambda & 0 \\ 0 & 0 & 1 - \lambda \end{pmatrix}$$

is also triangular, therefore the determinant of $N - \lambda I$ is the product of the entries along the main diagonal:

$$(2 - \lambda)(-4 - \lambda)(1 - \lambda),$$

and the roots of the characteristic equation of N are $\lambda = 2, -4$, and 1 .

If N is a triangular matrix, then the entries along its main diagonal are its eigenvalues.

Remark 5.2.1. If you add the algebraic multiplicity of all eigenvalues of a given matrix, it should be equal to the size of the matrix. The geometric multiplicity of an eigenvalue cannot be greater than its algebraic multiplicity.

5.3 Diagonalization

Definition 5.3.1. A square matrix is called *diagonalizable* if there exists an invertible matrix P so that $P^{-1}AP$ is diagonal.

Procedure for diagonalizing a matrix

1. Find the characteristic polynomial of the matrix A .
2. Find the roots to obtain the eigenvalues.
3. Repeat (a) and (b) for each eigenvalue λ of A :
 - (a) Form the augmented matrix to the equation $(A - \lambda I)\vec{v} = \vec{0}$ and bring it to a row-echelon form.
 - (b) Find a basis for the eigenspace corresponding to λ . That is find a basis for the solution space of $(A - \lambda I)\vec{v} = \vec{0}$.
4. Consider the collection $\mathcal{S} = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m\}$ of all basis vectors of the eigenspaces found in step 3.
 - (a) If m is less than the size of the matrix A , then A is not diagonalizable.
 - (b) If m is equal to the size of the matrix A , then A is diagonalizable, and the matrix P is the matrix whose columns are the vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ found in step 3, and

$$D = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}$$

where \vec{v}_1 corresponds to λ_1 , \vec{v}_2 corresponds to λ_2 , and so on.

We will look at the three examples we did in Section 5.2, and see whether the matrices A , B , and C are diagonalizable.

Example 5.3.1.

$$A = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}$$

is diagonalizable, because it has three basis vectors for all of its eigenspaces combined:

$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ are corresponding to $\lambda = 2$, and $\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$ is corresponding to $\lambda = 1$.

So

$$P = \begin{pmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

and

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note: Since we could have found another basis for the eigenspaces, this matrix P is not unique.

Example 5.3.2. The matrix

$$B = \begin{pmatrix} 5 & 0 & 4 \\ 0 & 3 & -1 \\ 0 & 0 & -2 \end{pmatrix}$$

is also diagonalizable, because we found 3 basis vectors for the eigenspaces combined.

Therefore

$$P = \begin{pmatrix} 1 & 0 & -20 \\ 0 & 1 & 7 \\ 0 & 0 & 35 \end{pmatrix}$$

and

$$D = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

Example 5.3.3. The matrix

$$C = \begin{pmatrix} 5 & -1 \\ 1 & 3 \end{pmatrix}.$$

is not diagonalizable, because it only has one basis vector for its eigenspace(s).

Example 5.3.4. If all eigenvalues are different, then the matrix is diagonalizable, because for each eigenvalue there will be one basis vector for the corresponding eigenspace. For example:

$$M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{pmatrix}$$

has eigenvalues $\lambda = 4, 2 \pm \sqrt{3}$. All eigenvalues are different, so A is diagonalizable. To find the matrix P , you will have to find a basis for each of the three eigenspaces. However, we already know the diagonal form will be:

$$D = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 + \sqrt{3} & 0 \\ 0 & 0 & 2 - \sqrt{3} \end{pmatrix}.$$

Example 5.3.5. The triangular matrix

$$N = \begin{pmatrix} 2 & 3 & -1 \\ 0 & -4 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

has eigenvalues $\lambda = 2, -4$ and 1 . The eigenvalues of N are all different, so N is diagonalizable, and D can be

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

To find the corresponding matrix P , to each eigenvalue you will have to find a corresponding eigenvector.

Example 5.3.6. The matrix

$$\begin{pmatrix} 0 & -2 \\ 3 & 0 \end{pmatrix}$$

has complex eigenvalues, $\lambda = \pm\sqrt{6}i$. In the diagonal form we would see these complex entries. Since the diagonal form is not a matrix over \mathbb{R} , we say this matrix is *not* diagonalizable over \mathbb{R} .

5.4 Computing Powers of a Matrix

There are numerous problems that require the computation of high powers of a matrix. If the matrix is diagonal, then this is easy.

Example 5.4.1. The 100th power of

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is

$$D^{100} = \begin{pmatrix} 2^{100} & 0 & 0 \\ 0 & 2^{100} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Suppose that a matrix A is not diagonal, but diagonalizable. That is

$$P^{-1}AP = D$$

for some diagonal matrix D and form some invertible matrix P . Multiply this equation by P from the left, and by P^{-1} from the right:

$$PP^{-1}APP^{-1} = PDP^{-1},$$

using that $PP^{-1} = I$, $P^{-1}P = I$ and $AI = A$, we get that

$$A = PDP^{-1}.$$

Now, let's take powers of A :

$$\begin{aligned} A^n &= (PDP^{-1})(PDP^{-1})(PDP^{-1}) \cdots (PDP^{-1})(PDP^{-1}) \\ &= PD(P^{-1}P)D(P^{-1}P)DP^{-1} \cdots PD(P^{-1}P)DP^{-1} \\ &= PDDD \cdots DDP^{-1} \\ &= PD^n P^{-1}. \end{aligned}$$

Therefore

$$A^n = PD^nP^{-1}.$$

Example 5.4.2. Let's calculate the 15th power of

$$A = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}.$$

We showed in Example 5.3.1 that A is diagonalizable with matrix

$$P = \begin{pmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix},$$

and then

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

So

$$\begin{aligned} A^{15} &= PD^{15}P^{-1} \\ &= \begin{pmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{15} \begin{pmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2^{15} & 0 & 0 \\ 0 & 2^{15} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 2 - 2^{15} & 0 & 2 - 2^{16} \\ 2^{15} - 1 & 2^{15} & 2^{15} - 1 \\ 2^{15} - 1 & 0 & 2^{16} - 1 \end{pmatrix}. \end{aligned}$$

For further applications you may see Section A.2 in the appendix.