Chapter 6

General Linear Transformations

6.1 Linear Transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$

We studied linear transformations in Section 4.2 and used the following definition to determine whether a transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ is linear.

Definition 6.1.1. A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear if the following two properties hold for all vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$ and every scalar c:

(a) $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$, and

(b) $T(c\vec{u}) = cT(\vec{u})$

Example 6.1.1. Using this theorem we can show for example that

1. $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (2x, x - y)$ is linear transformation.
2. $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (x^2, y)$ is not linear.

6.2 General Linear Transformations

Definition 6.2.1. Let V and W be two vector spaces, not necessarily a Euclidean space. A function f, which maps every vector from V to W is called a *map* or *transformation* from V to W. If $V = W$, then it can also be called *operator*.

Definition 6.2.2. Let V and W be two vector spaces. A transformation $T : V \to W$ is called a *linear transformation* if the following two properties hold for all vectors $\vec{u}$ and $\vec{v}$ in V and every scalar c:

(a) $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$, and

(b) $T(c\vec{u}) = cT(\vec{u})$

Example 6.2.1. Let V be the vector space of all real-valued functions that are differentiable, and let W be the vector space of all real-valued functions. Define $D : V \to W$ by

$$D(f) = f'$$

where f' is the derivative of f. We will show that this transformation is a linear transformation.

(a) Checking addition. Take two functions f and g from V, then

$$D(f + g) = (f + g)' = f' + g' = D(f) + D(g).$$

(b) Checking scalar multiplication. Take a function f form V and let c be a scalar. Then

$$D(cf) = (cf)' = cf' = cD(f).$$

Both of the properties of Definition6.2.2 are satisfied, so D is a linear transformation. Actually, this transformation is called the *differential operator*.

Example 6.2.2. Let V be the vector space of all real-valued functions that are integrable over the interval $[a, b]$. Let $W = \mathbb{R}$. Define $\mathfrak{I} : V \to W$ by

$$\mathfrak{I}(f) = \int_a^b f(x)\,dx.$$

Using calculus we can show that this transformation is linear:

(a) Checking addition. Take two functions f and g from V, then

$$\mathfrak{I}(f + g) = \int_a^b f(x) + g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx = \mathfrak{I}(f) + \mathfrak{I}(g).$$

(b) Checking scalar multiplication. Take a function f form V and let c be a scalar. Then

$$\mathfrak{I}(cf) = \int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx = c\mathfrak{I}(f).$$

Both of the properties are satisfied, so $\mathfrak{I}$ is a linear transformation.

Example 6.2.3. Consider the transformation $T : \mathcal{P}_2(t) \to \mathcal{P}_3(t)$ defined by

$$T\big(p(t)\big) = tp(t) + 1.$$

Since $p(t)$ is from $\mathcal{P}_2(t)$, the vector space of polynomials of degree 2 or less, we can write that $p(t) = at^2 + bt + c$. Let's see if this transformation is linear.

(a) Checking addition. Take two polynomials from $\mathcal{P}_2(t)$, say $p(t) = at^2 + bt + c$, and $q(t) = \alpha t^2 + \beta t + \gamma$. Then

$$\begin{aligned} T\big(p(t) + q(t)\big) &= T(at^2 + bt + c + \alpha t^2 + \beta t + \gamma) \\ &= T\big((a+\alpha)t^2 + (b+\beta)t + c + \gamma\big) \\ &= t\big((a+\alpha)t^2 + (b+\beta)t + c + \gamma\big) + 1 \\ &= (a+\alpha)t^3 + (b+\beta)t^2 + (c+\gamma) + 1. \end{aligned} \tag{6.2.1}$$

But

$$\begin{aligned} T\big(p(t)\big) + T\big(q(t)\big) &= T(at^2 + bt + c) + T(\alpha t^2 + \beta t + \gamma) \\ &= t(at^2 + bt + c) + 1 + t(\alpha t^2 + \beta t + \gamma) + 1 \\ &= (a+\alpha)t^3 + (b+\beta)t^2 + (c+\gamma) + 2. \end{aligned} \tag{6.2.2}$$

Since 6.2.1 and 6.2.2 are not equal, the first property of the definition 6.2.2 fails. Therefore this is not a linear transformation. (As a practice, show that the other property, $T\big(cp(t)\big) = cT\big(p(t)\big)$, also fails.)

Example 6.2.4. Let $\mathcal{P}_4(x)$ be the vector space of all real polynomials of degree 4 or less. Let $T : \mathcal{P}_4(x) \to \mathcal{P}_4(x)$ be given by

$$T = \frac{d^2}{dx^2} + 3\frac{d}{dx}.$$

First, let's just see how to calculate the image of a polynomial, say the image of $3x^4 - 5x^2 - 7x + 10$. That is, we have to find $T(3x^4 - 5x^2 - 7x + 10)$.

$$\begin{aligned} T(3x^4 - 5x^2 - 7x + 10) &= (3x^4 - 5x^2 - 7x + 10)'' + 3(3x^4 - 5x^2 - 7x + 10)' \\ &= 36x^2 - 10 + 3(12x^3 - 10x - 7) \\ &= 36x^3 + 36x^2 - 30x - 31. \end{aligned}$$

Is this transformation linear? We have to check the two properties given in the definition 6.2.2.

(a) Checking addition. Let $p(x)$ and $q(x)$ be two polynomials from $\mathcal{P}_4(x)$.

$$\begin{aligned} T(p(x)+q(x)) &= \frac{d^2}{dx^2}(p(x)+q(x)) + 3\frac{d}{dx}(p(x)+q(x)) \\ &= \frac{d^2}{dx^2}p(x) + \frac{d^2}{dx^2}q(x) + 3\left(\frac{d}{dx}p(x) + \frac{d}{dx}q(x)\right) \\ &= \frac{d^2}{dx^2}p(x) + 3\frac{d}{dx}p(x) + \frac{d^2}{dx^2}q(x) + 3\frac{d}{dx}q(x) \\ &= T(p(x)) + T(q(x)). \end{aligned}$$

(b) Checking scalar multiplication. Let $p(x)$ be a polynomial from $\mathcal{P}_4(x)$, and let k be any constant.

$$\begin{aligned} T(kp(x)) &= \frac{d^2}{dx^2}(kp(x)) + 3\frac{d}{dx}(kp(x)) \\ &= k\frac{d^2}{dx^2}p(x) + 3k\frac{d}{dx}p(x) \\ &= kT(p(x)). \end{aligned}$$

This shows that T is a linear transformation.

6.3 Matrix Representation of Linear Transformations

The matrix representation of a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ is the standard matrix of the transformation. But how can we find a matrix representation for a *general* linear transformation?

Example 6.3.1. We will use the transformation given in Example 6.2.4. Let $T : \mathcal{P}_4(x) \to \mathcal{P}_4(x)$ be given by

$$T = \frac{d^2}{dx^2} + 3\frac{d}{dx}.$$

The standard basis for $\mathcal{P}_4(x)$ is $\{x^4, x^3, x^2, x, 1\}$. First we calculate the image of each of these basis "vectors" under the transformation T.

$$\begin{aligned}
T(x^4) &= (x^4)'' + 3(x^4)' = 12x^2 + 12x^3 \\
T(x^3) &= (x^3)'' + 3(x^3)' = 6x + 9x^2 \\
T(x^2) &= (x^2)'' + 3(x^2)' = 2 + 6x \\
T(x) &= (x)'' + 3(x)' = 3 \\
T(1) &= (1)'' + 3(1)' = 0.
\end{aligned}$$

Now we have to "decode" these polynomials as follows

$$ax^4 + bx^3 + cx^2 + dx + e \text{ is represented by } \begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix}.$$

Using this, decode $T(x^4), T(x^3), T(x^2), T(x), T(1)$, the vectors we get are the columns of the standard matrix:

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 12 & 0 & 0 & 0 & 0 \\ 12 & 9 & 0 & 0 & 0 \\ 0 & 6 & 6 & 0 & 0 \\ 0 & 0 & 2 & 3 & 0 \end{pmatrix}.$$

6.4 Kernel and Range of Linear Transformations

Example 6.4.1. We continue working with the same example, the transformation $T : \mathcal{P}_4(x) \to \mathcal{P}_4(x)$ be given by

$$T = \frac{d^2}{dx^2} + 3\frac{d}{dx}.$$

Definition 6.4.1. The *kernel* of a transformation T form V to W is the set of all vectors from V which are mapped to the zero vector of W, that is

$$\operatorname{Ker}(T) = \{\vec{v} \in V : T(\vec{v}) = \vec{0}\}.$$

For transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ the kernel of a linear transformation is the null space of its standard matrix. For general linear transformations we can use the matrix representation, but we will have to "decode". So first we find the null space of the standard matrix A. The row-echelon form of A is

$$\text{rref}(A) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

so its null space is $\{(0,0,0,0,c) : c \in \mathbb{R}\}$, and a basis for this null space is $(0,0,0,0,1)$. Using the decoding again, the vector $(0,0,0,0,c)$ corresponds to $0 \cdot x^4 + 0 \cdot x^3 + 0 \cdot x^2 + 0 \cdot x + c \cdot 1 = c$, which is the constant polynomial c. So

$$\text{Ker}(T) = \{\text{all constant polynomials}\}.$$

The nullity(T) is the dimension of the kernel of T, therefore nullity$(T) = 1$.

Definition 6.4.2. The *range* of a transformation $T : V \to W$ is the set of all vectors $\vec{w} \in W$ for which there is a vector $\vec{v} \in V$ such that $T(\vec{v}) = \vec{w}$.

For linear transformation on Euclidean spaces, the range is equal to the column space of the standard matrix of the transformation. To find the range of a general linear transformation T, we can use its matrix representation.

So in our example, let's find the column space of its matrix representation. The first 4 columns of A form a basis for the column space, so

$$\text{column space} = \left\{ a \begin{pmatrix} 0 \\ 12 \\ 12 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 0 \\ 9 \\ 6 \\ 0 \end{pmatrix} + c \begin{pmatrix} 0 \\ 0 \\ 0 \\ 6 \\ 2 \end{pmatrix} + d \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 3 \end{pmatrix} : a, b, c, d \in \mathbb{R} \right\}.$$

A basis for the column space is:

$$\left\{ \begin{pmatrix} 0 \\ 12 \\ 12 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 9 \\ 6 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 6 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 3 \end{pmatrix} \right\}.$$

Remark 6.4.1. We could choose a simpler basis:

$$\left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 3 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

To get the range of T you have to "decode" the column space of A.

$$\text{Range}(T) = \left\{ a(12t^3 + 12t^2) + b(9t^2 + 6t) + c(6t + 2) + d \cdot 3 : a, b, c, d \in \mathbb{R} \right\}.$$

A basis for the range is:

$$\left\{ 12t^3 + 12t^2, 9t^2 + 6t, 6t + 2, 3 \right\}.$$