Appendix B
The Laplace Expansion For Determinant

B.1 First Minors and Cofactors; Row and Column Expansions

To each element a_{ij} in the determinant $|A| = |a_{ij}|_n$, there is associated a subdeterminant of order $(n-1)$ which is obtained from $|A|$ by deleting row i and column j. This subdeterminant is known as a first minor of $|A|$ and is denoted by $M_{ij} = D\left(A\left(i|j\right)\right)$. The first cofactor $(\mathcal{C}A)_{ij}$ is then defined as a signed first minor:

$$(\mathcal{C}A)_{ij} = (-1)^{i+j} M_{ij} = (-1)^{i+j} D\left(A\left(i|j\right)\right). \tag{B.1.1}$$

It is customary to omit the adjective *first* and to refer simply to minors and cofactors and it is convenient to regard M_{ij} and $(\mathcal{C}A)_{ij}$ as quantities which belong to a_{ij} in order to give meaning to the phrase "an element and its cofactor."

The expansion of $|A|$ by elements from row i and their cofactors is

$$|A| = D(A) = \sum_{j=1}^{n} a_{ij} (\mathcal{C}A)_{ij} = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} D\left(A\left(i|j\right)\right), \qquad 1 \le i \le n. \tag{B.1.2}$$

The expansion of $|A|$ by elements from column j and their cofactors is

$$|A| = D(A) = \sum_{i=1}^{n} a_{ij} (\mathcal{C}A)_{ij} = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} D\left(A\left(i|j\right)\right), \qquad 1 \le j \le n. \tag{B.1.3}$$

Since $(\mathcal{C}A)_{ij}$ belongs to but is independent of a_{ij}, another way to define $(\mathcal{C}A)_{ij}$ is

$$(\mathcal{C}A)_{ij} = \frac{\partial |A|}{\partial a_{ij}}. \tag{B.1.4}$$

B.2 Alien Cofactors; The Sum Formula

The theorem on alien cofactors states that

$$\sum_{j=1}^{n} a_{ij}(\mathcal{C}A)_{kj} = 0, \quad 1 \le i \le n, \quad 1 \le k \le n, \quad k \ne i. \tag{B.2.1}$$

The elements come from row i of $|A|$, but the cofactors belong to the elements in row k and are said to be alien to the elements. The identity is merely an expansion by elements from row k of the determinant in which row k = row i and which therefore zero.

The identity can be combined with the expansion formula for A with the aid of the Kronecker delta function δ_{ik} to form a single identity which may be called the sum formula for elements and cofactors:

$$\sum_{j=1}^{n} a_{ij}(\mathcal{C}A)_{kj} = \delta_{ik}|A|, \quad 1 \le i \le n, 1 \le k \le n. \tag{B.2.2}$$

It follows that

$$\sum_{j=1}^{n} (\mathcal{C}A)_{ij} C_j = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ D(A) \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \qquad 1 \le i \le n, \tag{B.2.3}$$

where C_j is column j of the matrix A, and the element $D(A)$ is in the row i of the column vector and all the other elements are zero. If $D(A) = 0$, then

$$\sum_{j=1}^{n} (\mathcal{C}A)_{ij} C_j = 0, \qquad 1 \le i \le n, \tag{B.2.4}$$

that is, the columns are linearly dependent. Conversely, if the columns are linearly dependent, then $D(A) = 0$.

B.3 Cramer's Formula

The set of equations

$$\sum_{j=1}^{n} a_{ij}x_j = b_i, \qquad 1 \le i \le n, \tag{B.3.1}$$

can be expressed in column vector notation as follows:

$$\sum_{\hat{j}=1}^{n} C_{\hat{j}} x_{\hat{j}} = B, \tag{B.3.2}$$

where

$$B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix}.$$

So

$$\begin{aligned}
\begin{vmatrix} C_1 & \cdots & C_{j-1} & B & C_{j+1} & \cdots & C_n \end{vmatrix} &= \begin{vmatrix} C_1 & \cdots & C_{j-1} & \sum_{\hat{j}=1}^{n} C_{\hat{j}} x_{\hat{j}} & C_{j+1} & \cdots & C_n \end{vmatrix} \\
&= \begin{vmatrix} C_1 & \cdots & C_{j-1} & x_1 C_1 & C_{j+1} & \cdots & C_n \end{vmatrix} \\
&+ \cdots \\
&+ \begin{vmatrix} C_1 & \cdots & C_{j-1} & x_{j-1} C_{j-1} & C_{j+1} & \cdots & C_n \end{vmatrix} \\
&+ \begin{vmatrix} C_1 & \cdots & C_{j-1} & x_j C_j & C_{j+1} & \cdots & C_n \end{vmatrix} \\
&+ \begin{vmatrix} C_1 & \cdots & C_{j-1} & x_{j+1} C_{j+1} & C_{j+1} & \cdots & C_n \end{vmatrix} \\
&+ \cdots \\
&+ \begin{vmatrix} C_1 & \cdots & C_{j-1} & x_n C_n & C_{j+1} & \cdots & C_n \end{vmatrix} \\
&= x_j \begin{vmatrix} C_1 & \cdots & C_{j-1} & C_j & C_{j+1} & \cdots & C_n \end{vmatrix} \\
&= x_j |A|.
\end{aligned} \tag{B.3.3}$$

If $|A| = |a_{ij}|_n \neq 0$, then the unique solution of the equations can also be expressed in column vector notation.

$$\begin{aligned} x_j &= \frac{1}{|A|} \begin{vmatrix} C_1 & \cdots & C_{j-1} & B & C_{j+1} & \cdots & C_n \end{vmatrix} \\ &= \frac{1}{|A|} \sum_{i=1}^{n} b_i (\mathcal{C}A)_{ij}. \end{aligned} \tag{B.3.4}$$