Chapter 1
Linear Equations

1.1 Systems of Linear Equations

A linear equation in the $n$ variables $x_1, x_2, \ldots, x_n$ is one that can be expressed in the form

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b$$

where $a_1, a_2, \ldots, a_n$ and $b$ are constants. The constants $a_1, a_2, \ldots, a_n$ are called the coefficients, $b$ is the constant term, and the variables $x_1, x_2, \ldots, x_n$ are also called unknowns. A solution of a linear equation $a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b$ is a sequence of $n$ numbers, so that the equation is satisfied when we substitute $x_1 = t_1, x_2 = t_2, \ldots, x_n = t_n$. We will also call it a solution vector and write it as $(t_1, t_2, \ldots, t_n)$. The collection of all solutions of the equation is called the solution set.

A finite set of linear equations is called a system of linear equations or a linear system. A sequence of numbers $(t_1, t_2, \ldots, t_n)$ is called a solution of the system if $x_1 = t_1, x_2 = t_2, \ldots, x_n = t_n$ is a solution of every equation in the system. If a system of equations has at least one solution (maybe infinitely many), then we say the system is consistent. If a system of equations has no solutions, then it is said to be inconsistent.

A linear equation in two unknowns, for example $2x_1 - 5x_2 = 3$, can be graphed in the Descartes coordinate system, and its graph is a straight line.

In the study of two equations

$$a_{11} x_1 + a_{12} x_2 = b_1$$
$$a_{21} x_1 + a_{22} x_2 = b_2$$

in two unknowns $x_1, x_2$ over $\mathbb{R}$ there are (assuming each equation determines a line) three possibilities:
• The two lines are distinct and not parallel. In this case they intersect in a point and there is a unique solution.

• The two lines are parallel. In this case there is no solution.

• The two lines coincide. In this case any solution of one equation is also a solution of the other, so there are infinitely many solutions.

**Example 1.1.1.** The system

\[
\begin{align*}
  x - y &= -10 \\
  3x + 5y &= 26
\end{align*}
\]

has one solution over \( \mathbb{R} \) (or over \( \mathbb{C} \)): \((-3, 7)\). That is \( x = -3 \) and \( y = 7 \) satisfies both of the equations. The system is consistent.

**Example 1.1.2.** The system

\[
\begin{align*}
  x - y &= -10 \\
  3x - 3y &= 3
\end{align*}
\]

has no solution over \( \mathbb{R} \) (or over \( \mathbb{C} \)). There is no such pair of numbers \((x, y)\), which satisfy both equations. The system is inconsistent.

**Example 1.1.3.** The system

\[
\begin{align*}
  x - y &= -10 \\
  3x - 3y &= -30
\end{align*}
\]

has infinitely many solutions over \( \mathbb{R} \), the solution set is \( \{(t, t + 10) : t \in \mathbb{R}\} \). That is, for every number \( t \), the pair \( x = t, y = t + 10 \) will satisfy both equations. The system is consistent.

**Example 1.1.4.** The system

\[
\begin{align*}
  x - y &= i - 2 \\
  5x - iy &= 15
\end{align*}
\]

has no solutions over \( \mathbb{R} \), but it has a solution over \( \mathbb{C} \), which is \((3 + i, 5)\). So it is always important to specify the field where you are looking for solutions. This system is inconsistent over \( \mathbb{R} \), but consistent over \( \mathbb{C} \).
1.2 Gauss-Jordan Elimination

If a system is in a so called triangular form, then you can solve it easily using back substitution.

Example 1.2.1. The system

\begin{align*}
x - 3y + 2z &= 17 \\
y + 5z &= 13 \\
z &= 3
\end{align*}

is triangular. The last equation gives you the value of \(z\) directly, \(z = 3\). Substituting this back to the second equation, you get the value of \(y\), \(y = -2\). Substituting \(z = 3\) and \(y = -2\) back into the first equation you can get \(x = 5\). The system is consistent, it has only one solution which is \((5, -2, 3)\).

Most of the systems are not in such a nice triangular form, but if we can bring it to a triangular form, then the solution is easy. How can we do that? If you multiply an equation by a nonzero number, does the solution change? If you swap two equations of the system, does the solution change? If you add a constant times an equation to another equation, does the solution change?

Example 1.2.2. We can bring the system

\begin{align*}
x + 2y - 3z &= -9 \\
y - 2z &= -4 \\
3x + 5y - 6z &= -20
\end{align*}

to a triangular form, if we first add \(-3\) times the first equation to the third equation \((R_3 \rightarrow R_3 - 3R_1)\)

\begin{align*}
x + 2y - 3z &= -9 \\
y - 2z &= -4 \\
-\cancel{y} + 3z &= 7,
\end{align*}
then we add the second equation to the third \((R_3 \rightarrow R_3 + R_2)\)
\[
\begin{align*}
x + 2y - 3z &= -9 \\
y - 2z &= -4 \\
z &= 3,
\end{align*}
\]
and then we can get the solution by back substitution: \((-4, 2, 3)\).

As we did this example, you may see that we have to keep track only the coefficients. Let’s look at a general linear system.
\[
\begin{align*}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m.
\end{align*}
\]
The coefficients of each variable aligned in columns
\[
\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}
\]
is called the \textit{coefficient matrix}, and if we also include the constant terms of the equations, then the matrix
\[
\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\
a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn} & b_m
\end{pmatrix}
\]
is called the \textit{augmented matrix} of the system. The \textit{size} of a matrix tells you how many rows and columns it has. The coefficient matrix above is an \(m \times n\) (read “m by n”) matrix. The augmented matrix above is \(m \times (n + 1)\). The following operations are allowed on the augmented matrix in order to get a solution.

\textbf{Definition 1.2.1.} Elementary row operations:
1. Multiply a row through by a nonzero number.

2. Interchange two rows.

3. Add a multiple of one row to another row.

**Example 1.2.3.** Let’s redo Example 1.2.2 using the augmented matrix. The augmented matrix of

\[
\begin{align*}
x + 2y - 3z &= -9 \\
y - 2z &= -4 \\
3x + 5y - 6z &= -20
\end{align*}
\]

is

\[
\begin{pmatrix}
1 & 2 & -3 & -9 \\
0 & 1 & -2 & -4 \\
3 & 5 & -6 & -20
\end{pmatrix}.
\]

We add \(-3\) times the first row to the third row \((R_3 \rightarrow R_3 - 3R_1)\), then the matrix changes to

\[
\begin{pmatrix}
1 & 2 & -3 & -9 \\
0 & 1 & -2 & -4 \\
0 & 1 & 3 & 7
\end{pmatrix},
\]

then we add the second row to the third one \((R_3 \rightarrow R_3 + R_2)\):

\[
\begin{pmatrix}
1 & 2 & -3 & -9 \\
0 & 1 & -2 & -4 \\
0 & 1 & 3 & 3
\end{pmatrix}.
\]

From here you can again use the equations given by this matrix. Now it is a triangular system, so use back substitution to find the solution.

The goal is to bring the augmented matrix to an easy-to-solve form, like

\[
\begin{pmatrix}
1 & 2 & -3 & -9 \\
0 & 1 & -2 & -4 \\
0 & 0 & 1 & 3
\end{pmatrix}.
\]

This matrix is in a *row echelon form*. To be of this form a matrix must have the following three properties:
1. If there are rows that consist entirely of zeroes, then they are all at the bottom of the matrix.

2. If a row does not consist entirely of zeroes, then the first nonzero number in the row must be a 1. We will call it leading 1.

3. In any two successive rows that do not consist entirely of zeroes, the leading 1 in the lower row occurs farther to the right than the leading 1 in the higher row.

Example 1.2.4. Here are some examples of matrices that are in row echelon form:

\[
\begin{pmatrix}
1 & \star & \star & \star \\
0 & 1 & \star & \star \\
0 & 0 & 1 & \star \\
0 & 0 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
1 & \star & \star & \star & \star \\
0 & 0 & 0 & 1 & \star \\
0 & 0 & 0 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
0 & 1 & \star & \star & \star \\
0 & 0 & 1 & \star & \star \\
0 & 0 & 0 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
1 & \star & \star & \star & \star \\
0 & 1 & \star & \star & \star \\
0 & 0 & 0 & 1 & \star \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

where \( \star \) can be any number.

If a matrix has in addition a fourth property:

4. Each column that contains a leading 1 has zeroes everywhere else.

then the matrix is said to be in reduced row echelon form (rref),

Example 1.2.5. Here are some examples of matrices that are in reduced row echelon form:

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
1 & \star & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 1 & 0 & \star \\
0 & 0 & 1 & \star \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
1 & 0 & 0 & \star \\
0 & 1 & 0 & \star \\
0 & 0 & 1 & \star \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

where \( \star \) can be any number.

Using elementary row operations you can always bring a matrix to a row echelon form. The procedure is called Gaussian elimination or Gaussian algorithm. We can work even more to bring the matrix into reduced row echelon form. This procedure is called Gauss-Jordan elimination. Here we note, that a matrix can have more than one row echelon form. But it can be shown that every matrix has a unique reduced row echelon form.
1.3 Examples

Note: This section is also available in txt format at
http://www.math.poly.edu/courses/ma2012/classnotes.phtml

that you can save (Save as ... File... choose a name, like class2.txt) and use with MATLAB. Open MATLAB, set your current directory to your working directory. Among the files you should see class2.txt. By clicking twice on this file you can open it. MATLAB will open the file using its own text editor. If you want to try something out, just cut and paste the command into the Command Window of the MATLAB. Hit the enter key to evaluate the command. MATLAB commands are written as:

>> command

Example 1.3.1. Solve the following system of linear equations over the real numbers:

\[
\begin{align*}
    x + y + 4z &= 15 \\
    2x + 4y - 3z &= 1 \\
    3x + 6y - 6z &= -3.
\end{align*}
\]

First write down the augmented matrix:

\[
\begin{bmatrix}
    1 & 1 & 4 & 15 \\
    2 & 4 & -3 & 1 \\
    3 & 6 & -6 & -3
\end{bmatrix}.
\]

Then use the Gaussian algorithm to bring the augmented matrix to row-echelon or reduced row-echelon form. Add $-2$ times the first row to the second:

\[
\begin{bmatrix}
    1 & 1 & 4 & 15 \\
    0 & 2 & -11 & -29 \\
    3 & 6 & -6 & -3
\end{bmatrix}.
\]

Notice that the third row can be divided by three, so divide the third row by 3:

\[
\begin{bmatrix}
    1 & 1 & 4 & 15 \\
    0 & 2 & -11 & -29 \\
    1 & 2 & -2 & -1
\end{bmatrix}.
\]
Subtract the first row from the third:

```matlab
>> M(3,:) = M(3,:) - M(1,:)
```

\[
M = \begin{bmatrix}
1 & 1 & 4 & 15 \\
0 & 2 & -11 & -29 \\
0 & 1 & -6 & -16
\end{bmatrix}.
\]

Switch the second and third rows:

```matlab
>> M([2,3],:) = M([3,2],:)
```

\[
M = \begin{bmatrix}
1 & 1 & 4 & 15 \\
0 & 1 & -6 & -16 \\
0 & 2 & -11 & -29
\end{bmatrix}.
\]

Add \(-2\) times the second row to the third one:

```matlab
>> M(3,:) = M(3,:) - 2*M(2,:)
```

\[
M = \begin{bmatrix}
1 & 1 & 4 & 15 \\
0 & 1 & -6 & -16 \\
0 & 0 & 1 & 3
\end{bmatrix}.
\]

Now you have the row-echelon form. From here you can get the solution by "back substitution": From the last row you have that \(z = 3\). Second row says that \(y - 6z = -16\), so \(y = 2\). First row says that \(x + y + 4z = 15\), so \(x = 1\).

Instead of the "back substitution" you can work more with the matrix to get the reduced row-echelon form. Subtract the second row from the first one:

```matlab
>> M(1,:) = M(1,:) - M(2,:)
```

\[
M = \begin{bmatrix}
1 & 0 & 10 & 31 \\
0 & 1 & -6 & -16 \\
0 & 0 & 1 & 3
\end{bmatrix}.
\]

Add \(-10\) times the third row the first row:

```matlab
>> M(1,:) = M(1,:) - 10*M(3,:)
```

\[
M = \begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & -6 & -16 \\
0 & 0 & 1 & 3
\end{bmatrix}.
\]
1.3 Examples

Add 6 times the third row to the second:

$$M(2,:) = M(2,:) + 6 * M(3,:)$$

$$M = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}.$$ 

Now you have the reduced row-echelon form. From this you can easily see the solutions. The first row says $x = 1$, the second row says $y = 2$, and the third row says $z = 3$. So the solution of the system is: $x = 1, y = 2, z = 3$. You can also write down the solution in the vector form: $(1, 2, 3)$. Notice that the first number is the value of $x$, the second is the value of $y$, and the third is the value of $z$. Keep the order of the variables!

MATLAB can give you the reduced row-echelon form in one step, let’s see how. Enter the augmented matrix (same as above):

$$M = \begin{bmatrix} 1 & 1 & 4 & 15 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -6 & -3 \end{bmatrix}.$$ 

Ask MATLAB to find the reduced row-echelon form:

$$\text{ans} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}.$$ 

That’s it. The solution of the system is $(1, 2, 3)$.

**Example 1.3.2.** Solve the system of linear equations over the real numbers:

\[
\begin{align*}
2y + 3z - v &= 1 \\
2x + 6y - 4z + 10v &= 8 \\
3x + 5y - 12z + 17v &= 7
\end{align*}
\]

Enter the augmented matrix:
>> M=[0 2 3 -1 1;2 6 -4 10 8;3 5 -12 17 7]

\[
M = \begin{bmatrix}
0 & 2 & 3 & -1 & 1 \\
2 & 6 & -4 & 10 & 8 \\
3 & 5 & -12 & 17 & 7 \\
\end{bmatrix}.
\]

Notice that the second row can be divided by 2, so divide the second row by 2:

>> M(2,:)=M(2,:)/2

\[
M = \begin{bmatrix}
0 & 2 & 3 & -1 & 1 \\
1 & 3 & -2 & 5 & 4 \\
3 & 5 & -12 & 17 & 7 \\
\end{bmatrix}.
\]

Since the second row starts with 1, which could be a convenient pivot point, switch
the first and second rows:

>> M([1,2],:)=M([2,1],:)

\[
M = \begin{bmatrix}
1 & 3 & -2 & 5 & 4 \\
0 & 2 & 3 & -1 & 1 \\
3 & 5 & -12 & 17 & 7 \\
\end{bmatrix}.
\]

Subtract 3 times the first row from the third:

>> M(3,:)=M(3,:)-3*M(1,:)

\[
M = \begin{bmatrix}
1 & 3 & -2 & 5 & 4 \\
0 & 2 & 3 & -1 & 1 \\
0 & -4 & -6 & 2 & -5 \\
\end{bmatrix}.
\]

Divide the second row by two, to get a 1 for you pivot point (this step is optional):

>> M(2,:)=M(2,:)/2

\[
M = \begin{bmatrix}
1.0000 & 3.0000 & -2.0000 & 5.0000 & 4.0000 \\
0 & 1.0000 & 1.5000 & -0.5000 & 0.5000 \\
0 & -4.0000 & -6.0000 & 2.0000 & -5.0000 \\
\end{bmatrix}.
\]

There are fractions in this matrix, and MATLAB shows you the numbers up to the
4th decimal place. If you would rather see the matrix in rational form, then type:

>> format rat

>> M

\[
M = \begin{bmatrix}
1 & 3 & -2 & 5 & 4 \\
0 & 1 & 3/2 & -1/2 & 1/2 \\
0 & -4 & -6 & 2 & -5 \\
\end{bmatrix}.
\]
However, we note here, that MATLAB uses rationales to APPROXIMATE the exact value. Sometimes it is a problem, please see Example 1 in the on-line MATLAB manual for further explanation on this.

Add 4 times the second row to the third:

\[
\begin{bmatrix}
1 & 3 & -2 & 5 & 4 \\
0 & 1 & 3/2 & -1/2 & 1/2 \\
0 & 0 & 0 & 0 & -3
\end{bmatrix}
\]

Look at the last row. That says: \(0 \times x + 0 \times y + 0 \times z + 0 \times v = -3\). There are no values for \((x, y, z, v)\) for which this can be true. So the system HAS NO SOLUTION. The system is inconsistent.

Again we could have used MATLAB build-in function to get the reduced row-echelon form in one step:

Enter the augmented matrix:

\[
\begin{bmatrix}
0 & 2 & 3 & -1 & 1; \\
2 & 6 & -4 & 10 & 8; \\
3 & 5 & -12 & 17 & 7
\end{bmatrix}
\]

Get the reduced row-echelon form:

\[
\begin{bmatrix}
1 & 0 & -13/2 & 13/2 & 0 \\
0 & 1 & 3/2 & -1/2 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

Look at the last row and conclude that the system is inconsistent.

**Example 1.3.3.** Find all real solutions \((p, q, r, s)\) of the system:

\[
\begin{align*}
p + 2r &= 0 \\
2p - 2q + 4r - 3s &= -1 \\
q + 3s &= 5 \\
2p + 8q + 4r + 15s &= 13.
\end{align*}
\]
Enter the augmented matrix:

\[
M = \begin{bmatrix}
1 & 0 & 2 & 0 & 0 \\
2 & -2 & 4 & -3 & -1 \\
0 & 1 & 0 & 3 & 5 \\
2 & 8 & 4 & 15 & 13
\end{bmatrix}
\]

Bring to reduced row-echelon form:

\[
\text{ans} = \begin{bmatrix}
1 & 0 & 2 & 0 & 0 \\
0 & 1 & 0 & 0 & -4 \\
0 & 0 & 0 & 1 & 3 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

This means: \( p + 2r = 0 \), \( q = -4 \) and \( s = 3 \). There are only three equations, but we have 4 unknowns. That means the fourth unknown is free, that can be anything.

How to give the solutions?

The unknown that has a leading 1 in its column is determined by the row of its leading 1. The unknown that has no leading 1 in its column is a free variable.

That is, \( p \) is determined by the first row: \( p + 2r = 0 \), or if you rearrange this: \( p = -2r \).

\( q \) has a leading 1, so it is determined by the second row: \( q = -4 \).

\( r \) has no leading 1, so it is a free variable.

\( s \) has a leading 1, so it is determined by the third row: \( s = 3 \).

So the solutions are: \( p = -2r, q = -4, r, s = 3 \). Writing this down with the vector notation, the solutions are: \((-2r, -4, r, 3)\). The system has infinitely many solutions.

**Example 1.3.4.** For which value(s) of the constant \( k \) does the system

\[
x + (k - 4)y = k + 3 \\
-kx + (2k - 3)y = 2
\]

have
(a) no solution

(b) exactly 1 solution

(c) infinitely many solutions

over the field of real numbers?

First we have to teach MATLAB that “$k$” is a parameter (a symbolic variable):

```matlab
>> syms k
>> M=[1 k-4 k+3;-k 2*k-3 2]
M =
1, k - 4, k + 3
-k, 2 * k - 3, 2
```

Add $k$ times the first row to the second row:

```matlab
>> M(2,:)=M(2,:)+k*M(1,:)
```

```
M =
1, k - 4, k + 3
0, 2 * k - 3 + k * (k - 4) 2 + k * (k + 3)
```

Factor the nonzero terms in the last row:

```matlab
>> M(2,2)=factor(M(2,2));
>> M(2,3)=factor(M(2,3))
```

```
M =
1, k - 4, k + 3
0, (k + 1) * (k - 3) (k + 2) * (k + 1)
```

(a) The system has no solution if the last row becomes: 0,0,nonzero. So if $(k + 1) * (k - 3) = 0$ but $(k + 2) * (k + 1)$ is not zero. This happens if $k = 3$.

(b) The system has exactly one solution, if the second row also has a leading one, that is $(k + 1) * (k - 3)$ is not zero. Therefore the system has exactly one solution if $k$ is neither equal to $-1$ nor equal to 3. Notice that $(k + 2) * (k + 1)$ can be anything, zero or not zero, you do still get exactly one solution.

(c) The system has infinitely many solutions, if the second row has no leading and the system has solution(s). That is when both $(k + 1) * (k - 3)$ and $(k + 2) * (k + 1)$ are equal to 0. This happens when $k = -1$. 

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Remark 1.3.1. I do NOT suggest using the command \texttt{rref}(M) for this problem. Try it, and see what happens! How would you answer the questions from that form? Why don’t you get the same answer for part (c)? You will not be able to answer part (c) correctly, because MATLAB (and most of the calculators) will divide the second row by $(k+1)$. You know that you cannot divide by $(k+1)$ if that is zero, MATLAB just assumes that is not zero. Which is not correct.

1.4 The $\mathbb{Z}_p$ field

Definition 1.4.1. Two integers $a$ and $b$ are said to be congruent modulo $p$, written

$$a \equiv b \pmod{p},$$

if $p$ divides $b-a$.

Example 1.4.1.

- $-1 \equiv 1 \pmod{2}$
- $22 \equiv 1 \pmod{3}$
- $12 \equiv 2 \pmod{5}$
- $-12 \equiv 3 \pmod{5}$

Definition 1.4.2. $\mathbb{Z}_p$ denotes the residue classes modulo $p$, that is

$$\mathbb{Z}_p = \{0, 1, 2, \ldots, p-1\}.$$ 

Example 1.4.2.

- $\mathbb{Z}_2 = \{0, 1\}$
- $\mathbb{Z}_3 = \{0, 1, 2\}$
- $\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$

You can add, subtract and multiply numbers in $\mathbb{Z}_p$, but no division!
Example 1.4.3.

- $1 + 1 \equiv 0 \pmod{2}$
- $2 + 2 \equiv 1 \pmod{3}$
- $3 + 4 \equiv 2 \pmod{5}$
- $3 - 4 \equiv 4 \pmod{5}$
- $2 \cdot 2 \equiv 1 \pmod{3}$
- $4 \cdot 4 \equiv 1 \pmod{5}$
- $3 \cdot 4 \equiv 2 \pmod{5}$
- $6 \cdot 3 \equiv 4 \pmod{7}$
- $5 \cdot 3 \equiv 1 \pmod{7}$

1.5 Systems of Linear Equations in $\mathbb{Z}_p$

Example 1.5.1. Solve the following system of linear equations in $\mathbb{Z}_3$:

\[
\begin{align*}
\begin{cases}
x + y + 2z &= 0 \\
2x + 2z &= 1 \\
x + 2y &= 2.
\end{cases}
\end{align*}
\]

The augmented matrix of this system is:

\[
\begin{pmatrix}
1 & 1 & 2 & | & 0 \\
2 & 0 & 2 & | & 1 \\
1 & 2 & 0 & | & 2
\end{pmatrix}
\]

We use the Gaussian algorithm in $\mathbb{Z}_3$ to reduce the matrix to a row-echelon form. Add the first row to the second (mod 3):

\[
\begin{pmatrix}
1 & 1 & 2 & | & 0 \\
0 & 1 & 1 & | & 1 \\
1 & 2 & 0 & | & 2
\end{pmatrix}
\]

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subtract the first row from the third \((\text{mod } 3)\):
\[
\begin{pmatrix}
1 & 1 & 2 & 0 \\
0 & 1 & 1 & 1 \\
0 & 1 & 1 & 2
\end{pmatrix},
\]
subtract the second row from the third:
\[
\begin{pmatrix}
1 & 1 & 2 & 0 \\
0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]
The last row says: \(0 \cdot x + 0 \cdot y + 0 \cdot z = 1\), which is not possible, therefore the system is inconsistent and it has no solution in \(\mathbb{Z}_3\).

**Example 1.5.2.** Solve the following system in \(\mathbb{Z}_2\):
\[
\begin{align*}
x_1 + x_2 &= 1 \\
x_2 + x_3 + x_4 &= 0 \\
x_1 + x_4 &= 0 \\
x_1 + x_2 + x_3 &= 0.
\end{align*}
\]
The augmented matrix is
\[
\begin{pmatrix}
1 & 1 & 0 & 0 & 1 \\
0 & 1 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & 0
\end{pmatrix}.
\]
Using row operations in \(\mathbb{Z}_2\), add the first row to the third and fourth rows:
\[
\begin{pmatrix}
1 & 1 & 0 & 0 & 1 \\
0 & 1 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1
\end{pmatrix},
\]
add the second row to the third row:
\[
\begin{pmatrix}
1 & 1 & 0 & 0 & 1 \\
0 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1
\end{pmatrix},
\]
\[\]
add the third row to the fourth:

\[
\begin{pmatrix}
1 & 1 & 0 & 0 & 1 \\
0 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

This is now in row-echelon form. Notice that there are only 3 leading 1, so one of the four unknowns will be free. We can use back-substitution to solve:

The fourth column has no leading one, that means \(x_4\) is a free variable. That means \(x_4\) is either 0 or 1, since there are the only elements in \(\mathbb{Z}_2\).

The first row says: \(x_1 + x_2 = 1\). By solving for \(x_1\): \(x_1 = 1 + x_2\) (remember we are calculating in \(\mathbb{Z}_2\)).

The second row says: \(x_2 + x_3 + x_4 = 0\). By solving for \(x_2\): \(x_2 = x_3 + x_4\).

The third row says: \(x_3 = 1\).

So if \(x_4 = 0\), then the solution is \((x_1, x_2, x_3, x_4) = (0, 1, 1, 0)\); and if \(x_4 = 1\), then the solution is \((x_1, x_2, x_3, x_4) = (1, 0, 1, 1)\). The system has these two solutions in \(\mathbb{Z}_2\).