Chapter 2
Matrix Algebra

2.1 Matrix Arithmetic

Definition 2.1.1. A matrix is a rectangular array of numbers. We will always denote matrices by capital letters. The numbers in the array are called entries in the matrix. If we want to refer to the entry of matrix A standing in the $i$th row and $j$th column, then we write $(A)_{ij}$ or $a_{ij}$. If a matrix has $m$ rows and $n$ columns, then the matrix is said be of size $m \times n$. The entries of a matrix can be real numbers, then we say the matrix is over $\mathbb{R}$. If the entries are complex numbers, then the matrix is said to be over $\mathbb{C}$. If we want to refer to the entry of matrix $A$ that is in the $i$th row and $j$th column, we write $(A)_{ij}$ or $a_{ij}$. A matrix which has only one row is also called a row matrix. A matrix which has only one column is also called a column matrix or vector. A matrix with $n$ rows and $n$ columns is called a square matrix of order $n$, and the entries $a_{11}, a_{22}, \ldots, a_{nn}$ are said to be on the main diagonal of the matrix.

Example 2.1.1.

\[
A = \begin{pmatrix}
2 & 4 & -1 & 0 \\
3 & -3 & 1 & 2 \\
0 & 2 & 1 & 0
\end{pmatrix}, \quad B = \begin{pmatrix}
2 & 0 & 1 & -2 & 3
\end{pmatrix}, \\
C = \begin{pmatrix}
-1 \\
0 \\
2 \\
7
\end{pmatrix}, \quad D = \begin{pmatrix}
2 & 3 \\
-1 & 7
\end{pmatrix}, \quad E = \begin{pmatrix}
5
\end{pmatrix}.
\]

Matrix $A$ is of size $3 \times 4$. Matrix $B$ is of size $1 \times 5$, it is also called a row matrix. Matrix $C$ is of size $4 \times 1$, it is also called a column matrix or vector. Matrix $D$ is a $2 \times 2$ matrix; since it has the same number of rows and columns it is called a square
matrix of order 2. Its main diagonal consists of the numbers 2 and 7. Matrix $E$ is of size $1 \times 1$; every number can be considered a $1 \times 1$ matrix.

**Definition 2.1.2.** The $m \times n$ zero matrix is the $m \times n$ matrix whose entries are all zeroes.

**Example 2.1.2.** Some examples for zero matrices:

$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, ...

**Definition 2.1.3.** A square matrix whose entries along its main diagonal are all 1’s and whose all other entries are 0’s is called an identity matrix. The $n \times n$ identity matrix is denoted by $I_n$.

**Example 2.1.3.** The following are examples for identity matrices:

$I_1 = \begin{pmatrix} 1 \end{pmatrix}$, $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, ...

**Definition 2.1.4.** Two matrices are *equal* if they have the same size and their corresponding entries are equal.

**Definition 2.1.5.** If two matrices, $A$ and $B$, are of the same size, then the *sum* $A + B$ is the matrix obtained by adding the entries of $B$ to the corresponding entries of $A$.

**Definition 2.1.6.** If two matrices, $A$ and $B$, are of the same size, then the *difference* $A - B$ is the matrix obtained by subtracting the entries of $B$ from the corresponding entries of $A$.

**Example 2.1.4.** Consider the matrices

$A = \begin{pmatrix} 2 & 1 & 0 & -1 \\ 3 & -2 & 2 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 2 & 3 & 1 \\ -1 & -5 & 3 & 7 \end{pmatrix}$. 
Both matrices have the same size, $2 \times 4$, so the sum and difference are defined.

\[
A + B = \begin{pmatrix}
2 & 3 & 3 & 0 \\
-2 & -7 & 5 & 7
\end{pmatrix},
\]

and

\[
A - B = \begin{pmatrix}
2 & -1 & -3 & -2 \\
4 & 3 & -1 & -7
\end{pmatrix}.
\]

**Definition 2.1.7.** If $A$ is a matrix and $c$ is a scalar, then the product $cA$ is the matrix obtained by multiplying each entry of the matrix $A$ by $c$.

**Example 2.1.5.** Let

\[
A = \begin{pmatrix}
2 & 1 & 0 & -1 \\
3 & -2 & 2 & 0
\end{pmatrix}.
\]

Then

\[
-5A = \begin{pmatrix}
-10 & -5 & 0 & 5 \\
-15 & 10 & -10 & 0
\end{pmatrix}.
\]

**Theorem 2.1.1 (Properties of Matrix Arithmetic I).** Assuming that the sizes of the matrices are such that the indicated operations can be performed, the following rules of matrix arithmetic are valid. Here $A$, $B$ and $C$ denote matrices, $0$ is a zero matrix, and $a$ and $b$ are scalars.

1. $A + B = B + A$
   
   Commutative law for addition

2. $A + (B + C) = (A + B) + C$
   
   Associative law for addition

3. $a(B + C) = aB + aC = (B + C)a$

4. $a(B - C) = aB - aC = (B - C)a$

5. $(a + b)C = aC + bC = C(a + b)$

6. $(a - b)C = aC - bC = C(a - b)$

7. $a(bC) = (ab)C$
8. \( A + 0 = 0 + A = A \)

9. \( A - A = 0 \)

10. \( 0 - A = -A \)

**Definition 2.1.8.** If \( A \) is an \( m \times r \) matrix and \( B \) is an \( r \times n \) matrix, then the **matrix product** \( AB \) is the \( m \times n \) matrix whose entries are obtained as follows:

For the entry \((AB)_{ij}\) choose the \( i \)th row of matrix \( A \) and the \( j \)th column of matrix \( B \), and multiply the corresponding entries of this row and column together and then add up the resulting products. (See Figure 2.1 for the sizes.)

\[
(A)_{ik} \quad (B)_{kj}
\]

With sum notation we can write

\[
(AB)_{ij} = \sum_{k=1}^{r} (A)_{ik}(B)_{kj}.
\]

**Remark 2.1.1.** It is easy to calculate \( AB \) if you write the two matrices arranged like this:

\[
\begin{array}{c|c}
A & B \\
\hline
A & AB
\end{array}
\]
For example,

\[ \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \\ b_{41} & b_{42} & b_{43} \end{bmatrix} \]

\[ \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \\ a_{51} & a_{52} & a_{53} & a_{54} \end{bmatrix} \]

\[ \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \\ c_{41} & c_{42} & c_{43} \\ c_{51} & c_{52} & c_{53} \end{bmatrix} \]

where for example \( c_{32} = \text{row}_3(A) \text{col}_2(B) \) is given by

\[
c_{32} = a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} + a_{34}b_{42}.
\]

**Example 2.1.6.** Consider the matrices

\[
A = \begin{pmatrix} 2 & 1 & 0 & -1 \\ 3 & -2 & 2 & 0 \end{pmatrix}_{2 \times 4}, \quad \text{and}
B = \begin{pmatrix} 1 & 0 & -1 \\ -2 & 2 & 0 \\ 0 & 3 & 2 \\ -3 & 5 & 2 \end{pmatrix}_{4 \times 3}.
\]

Then

\[
AB = \begin{pmatrix} 3 & -3 & -4 \\ 7 & 2 & 1 \end{pmatrix}_{2 \times 3},
\]

but \( BA \) is not defined since the number of columns in \( B \) and the number of rows in \( A \) are not the same.

**Example 2.1.7.** If

\[
A = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \text{and} \quad B = \begin{pmatrix} 5 & -3 \end{pmatrix},
\]

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\[
AB = \begin{pmatrix} 10 & -6 \\ 5 & -3 \end{pmatrix} \quad \text{and} \quad BA = \begin{pmatrix} 7 \end{pmatrix}.
\]

**Definition 2.1.9.** For a positive integer \( p \) and a square matrix \( A \) we define the \( p \)-th power of \( A \) by

\[
A^p = \underbrace{AAA \cdots A}_p,
\]

and we also define \( A^0 = I \).

Would \( A^p \) be defined for a non-square matrix?

**Theorem 2.1.2 (Properties of Matrix Arithmetic II).** Assuming that the sizes of the matrices are such that the indicated operations can be performed, the following rules of matrix arithmetic are valid. Here \( A, B \) and \( C \) denote matrices, 0 is a zero matrix, \( a \) is a scalar, and \( r \) and \( s \) are positive integers.

1. \( A(BC) = (AB)C \)
   
   Associative law for multiplication

12. \( A(B + C) = AB + AC \)
   
   Left distributive law

13. \( (B + C)A = BA + CA \)
   
   Right distributive law

14. \( A(B - C) = AB - AC \)

15. \( (B - C)A = BA - CA \)

16. \( a(BC) = (aB)C = B(aC) \)

17. \( A0 = 0A = 0 \)

18. \( AI = IA = A \)

19. \( A^r A^s = A^{r+s} \)

20. \( (A^r)^s = A^{rs} \)
21. $A^0 = I$

**Definition 2.1.10.** If $A$ is any $m \times n$ matrix, then the *transpose of $A$*, denoted by $A^T$, is defined to be the $n \times m$ matrix, that is obtained by interchanging the rows and columns of $A$.

**Example 2.1.8.** If

$$A = \begin{pmatrix} 2 & 1 & 0 & -1 \\ 3 & -2 & 2 & 0 \end{pmatrix}_{2 \times 4}$$

then

$$A^T = \begin{pmatrix} 2 & 3 \\ 1 & -2 \\ 0 & 2 \\ -1 & 0 \end{pmatrix}_{4 \times 2}$$

**Theorem 2.1.3 (Properties of Matrix Arithmetic III).** Assuming that the sizes of the matrices are such that the indicated operations can be performed, the following rules of matrix arithmetic are valid. Here $A$ and $B$ denote matrices, and $a$ is a scalar.

22. $(A + B)^T = A^T + B^T$

23. $(A - B)^T = A^T - B^T$

24. $(AB)^T = B^T A^T$

25. $(A^T)^T = A$

26. $(aB)^T = aB^T$

**Definition 2.1.11.** If $A$ is a square matrix, then the *trace of $A$*, denoted by $\text{tr}(A)$, is defined to be the sum of the entries along the main diagonal of $A$.

**Example 2.1.9.** The trace of

$$A = \begin{pmatrix} 2 & 1 & 0 & -1 \\ 3 & 2 & 0 & -1 \end{pmatrix}$$

is undefined, since $A$ is not a square matrix.

**Example 2.1.10.** Consider the square matrix

$$D = \begin{pmatrix} 2 & 3 \\ -1 & 7 \end{pmatrix}.$$ 

Then $\text{tr}(D) = 2 + 7 = 9$. 

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Theorem 2.1.4 (Properties of Matrix Arithmetic IV). Assuming that the sizes of the matrices are such that the indicated operations can be performed, the following rules of matrix arithmetic are valid. Here $A$ and $B$ denote matrices, and $c$ is a scalar.

27. $\text{tr}(A + B) = \text{tr}(A) + \text{tr}(B)$

28. $\text{tr}(A - B) = \text{tr}(A) - \text{tr}(B)$

29. $\text{tr}(cB) = c \cdot \text{tr}(B)$

30. $\text{tr}(AB) = \text{tr}(BA)$

Proof. Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix},$$

then $\text{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn}$ and $\text{tr}(B) = b_{11} + b_{11} + \cdots + b_{nn}$.

Since

$$A + B = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2n} + b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} + b_{n1} & a_{n2} + b_{n2} & \cdots & a_{nn} + b_{nn} \end{pmatrix},$$

$$\text{tr}(A + B) = a_{11} + b_{11} + a_{22} + b_{22} + \cdots + a_{nn} + b_{nn} = a_{11} + a_{22} + \cdots + a_{nn} + b_{11} + b_{22} + \cdots + b_{nn} = \text{tr}(A) + \text{tr}(B).$$

This proves that $\text{tr}(A + B) = \text{tr}(A) + \text{tr}(B)$. Similarly you can prove that $\text{tr}(A - B) = \text{tr}(A) - \text{tr}(B)$.

Since

$$cA = \begin{pmatrix} ca_{11} & ca_{12} & \cdots & ca_{1n} \\ ca_{21} & ca_{22} & \cdots & ca_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ca_{n1} & ca_{n2} & \cdots & ca_{nn} \end{pmatrix},$$
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\[ \text{tr}(cA) = ca_{11} + ca_{22} + \cdots + ca_{nn} \]
\[ = c(a_{11} + a_{22} + \cdots + a_{nn}) \]
\[ = c \text{tr}(A). \]

This proves that \( \text{tr}(cA) = c \text{tr}(A). \)

If we calculate the product matrix \( AB \), then the entries along the main diagonal are:

\[
(AB)_{11} = (a_{11} + \cdots + a_{1n})(b_{11} + \cdots + b_{n1}) \\
(AB)_{22} = (a_{21} + \cdots + a_{2n})(b_{12} + \cdots + b_{n2}) \\
\vdots \\
(AB)_{nn} = (a_{n1} + \cdots + a_{nn})(b_{1n} + \cdots + b_{nn}),
\]

so

\[
\text{tr}(AB) = (a_{11} + \cdots + a_{1n})(b_{11} + \cdots + b_{n1}) + (a_{21} + \cdots + a_{2n})(b_{12} + \cdots + b_{n2}) \\
+ \cdots + (a_{n1} + \cdots + a_{nn})(b_{1n} + \cdots + b_{nn}).
\] (2.1.1)

The entries along the main diagonal of the product \( BA \) are:

\[
(BA)_{11} = (b_{11} + \cdots + b_{1n})(a_{11} + \cdots + a_{n1}) \\
(BA)_{22} = (b_{21} + \cdots + b_{2n})(a_{12} + \cdots + a_{n2}) \\
\vdots \\
(BA)_{nn} = (b_{n1} + \cdots + b_{nn})(a_{1n} + \cdots + a_{nn}),
\]

so

\[
\text{tr}(BA) = (b_{11} + \cdots + b_{1n})(a_{11} + \cdots + a_{n1}) + (b_{21} + \cdots + b_{2n})(a_{12} + \cdots + a_{n2}) \\
+ \cdots + (b_{n1} + \cdots + b_{nn})(a_{1n} + \cdots + a_{nn}).
\] (2.1.2)

If you expand 2.1.1 and 2.1.2, you can see that they have exactly the same terms, so

\[ \text{tr}(AB) = \text{tr}(BA). \]

\[ \square \]
Finally, we would like to emphasize two very important things about matrix calculations.

Remark 2.1.2. For matrices the cancellation law usually does not hold!
Consider the matrices

\[ A = \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 3 & 4 \end{pmatrix}, \]
\[ C = \begin{pmatrix} 2 & 5 \\ 3 & 4 \end{pmatrix}, \quad \text{and} \quad D = \begin{pmatrix} 3 & 7 \\ 0 & 0 \end{pmatrix}. \]

You can calculate that

\[ AB = AC = \begin{pmatrix} 3 & 4 \\ 6 & 8 \end{pmatrix}, \quad \text{and} \quad AD = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \]

That is, although \( AB = AC \) and \( A \neq 0 \), but it is incorrect to cancel \( A \) from both sides. We cannot conclude that \( B = C \).
Moreover, although \( AD = 0 \), yet \( A \neq 0 \) and \( D \neq 0 \).

Remark 2.1.3. The matrix multiplication is not commutative! Using the matrices from the previous remark, verify that \( AB \neq BA \).

## 2.2 Elementary Matrices

Elementary matrices are the simplest of all invertible matrices. We shall see that they are the building blocks from which the invertible matrices are constructed. Here is the definition.

**Definition 2.2.1.** A matrix that results from the identity matrix by applying a single elementary row operation (see Definition 1.2.1) is called an **elementary matrix**. An elementary matrix is always a square matrix. There are three kinds.

**Scale** The matrix \( E = \text{Scale}(I, i, c) \) is an elementary matrix for \( i = 1, 2, \ldots, m \) and \( c \neq 0 \). It differs from the \( m \times m \) identity matrix \( I = I_m \) in that \((E)_{ii} = c\) rather than 1.
Swap The matrix $E = \text{Swap}(I, i, j)$ is an elementary matrix for $i, j = 1, 2, \ldots, m, i \neq j$. It differs from the identity matrix in that
\[
(E)_{ii} = 0 \quad (E)_{ij} = 1 \\
(E)_{ji} = 1 \quad (E)_{jj} = 0.
\]

Shear The matrix $E = \text{Shear}(I, i, j, c)$ is an elementary matrix for $i, j = 1, 2, \ldots, m, i \neq j$. It differs from the identity matrix in that
\[
(E)_{ij} = c.
\]

Example 2.2.1. The matrices
\[
E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]
are examples for elementary matrices. $E_1 = \text{Scale}(I, 3, -4)$ is a scale, $E_2 = \text{Swap}(I, 1, 2)$, and $E_3 = \text{Shear}(I, 1, 3, 2)$.

What happens if we multiply a $3 \times 3$ matrix by these elementary matrixes from the left. (Remember: matrix multiplication is not commutative!) Let
\[
A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},
\]
then
\[
E_1A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ -4a_{31} & -4a_{32} & -4a_{33} \end{pmatrix}, \quad E_2A = \begin{pmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad \text{and}
\]
\[
E_3A = \begin{pmatrix} a_{11} + 2a_{31} & a_{12} + 2a_{32} & a_{13} + 2a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.
\]

Theorem 2.2.1. The matrix $EA$ that results by multiplying a matrix $A$ on the left by an elementary matrix $E$ is the same as the matrix that results by applying the corresponding elementary row operation to $A$. 

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2.3 Matrix Inverse

Definition 2.3.1. If $A$ is a square matrix, and if $B$ of the same size can be found such that $AB = I$ and $BA = I$, then $A$ is said to be invertible and $B$ is called an inverse of $A$. If no such matrix $B$ can be found, then $A$ is said to be singular or non-invertible. The inverse of $A$ is denoted by $A^{-1}$.

Example 2.3.1. The matrix $A = \begin{pmatrix} 2 & 3 & 0 \\ 1 & -2 & 5 \end{pmatrix}$ has no inverse, because it is not a square matrix.

Example 2.3.2. A diagonal matrix with nonzero entries along its diagonal is invertible, and its inverse is also diagonal whose diagonal entries are the reciprocals of the original diagonal entries. For example

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & -4 \end{pmatrix}^{-1} = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1/4 \end{pmatrix}.$$

Example 2.3.3. The matrix $B = \begin{pmatrix} 1/2 & -3/2 \\ 0 & 1 \end{pmatrix}$ is an inverse of $A = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$, because

$$AB = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1/2 & -3/2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and

$$BA = \begin{pmatrix} 1/2 & -3/2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Example 2.3.4. The matrix

$$\begin{pmatrix} 0 & 0 & 0 \\ -3 & 1 & 4 \\ 2 & 5 & 7 \end{pmatrix}$$

is singular, it has no inverse, because the first row of the matrix product $AB$ is always a zero row, so $AB$ cannot give the identity matrix.
Example 2.3.5. Every elementary matrix is invertible, and the inverse is also an elementary matrix. The inverse of \( \text{Scale}(I, i, c) \) is \( \text{Scale}(I, i, \frac{1}{c}) \). The inverse of \( \text{Swap}(I, i, j) \) is \( \text{Swap}(I, i, j) \). The inverse of \( \text{Shear}(I, i, j, c) \) is \( \text{Shear}(I, i, j, -c) \).

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -4
\end{pmatrix}^{-1} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1/4
\end{pmatrix},
\]

\[
\begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}^{-1} = \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

\[
\begin{pmatrix}
1 & 0 & 2 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}^{-1} = \begin{pmatrix}
1 & 0 & -2 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

Example 2.3.6. Let \( A \) and \( B \) be two invertible matrices of the same size. Then

\[
(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}B = B^{-1}B = I,
\]

and

\[
(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I.
\]

This shows that the product \( AB \) is invertible, and

\[
(AB)^{-1} = B^{-1}A^{-1}.
\]

Theorem 2.3.1 (Properties of Matrix Arithmetic V). If \( A \) and \( B \) are invertible matrices, then:

31. \((A^{-1})^{-1} = A\)

32. \((A^n)^{-1} = (A^{-1})^n\)

33. \((aA)^{-1} = \frac{1}{a}A^{-1}\), for any nonzero scalar \( a \).

34. \((AB)^{-1} = B^{-1}A^{-1}\).

35. \((A^T)^{-1} = (A^{-1})^T\).


\section{Method for Finding the Inverse}

Using row operations to find \( A^{-1} \): Construct the matrix

\[ [A \mid I] \]

and apply row operations until you can get the identity matrix on the left side. If that is possible, then the matrix on the right side will be the inverse of \( A \). So the final matrix will be

\[ [I \mid A^{-1}] \]

If we cannot get \( I \) on the left hand side using elementary row operations, then \( A \) is non-invertible.

\textbf{Proof.} Let \( A \) be a square matrix. Every elementary row operation corresponds to a multiplications by an elementary matrix (scale, swap or shear) from the left. We do the row operations on \([A \mid I]\) until we get the row reduced echelon form (rref) of \( A \) on the left side.

\[ E_k \cdots E_3 E_2 E_1 [A \mid I] = [E_k \cdots E_2 E_1 A \mid E_k \cdots E_3 E_2 E_1 I] = [\text{rref} \mid E_k \cdots E_3 E_2 E_1]. \]

There are two possibilities:

1. The \text{rref} of \( A \) has a zero row. In this case the matrix \( A \) is not invertible.

2. The \text{rref} of \( A \) is the identity. That is \( E_k \cdots E_2 E_1 A = I \), which means that \( E_k \cdots E_2 E_1 \) is an inverse of \( A \). Therefore

\[ E_k \cdots E_3 E_2 E_1 [A \mid I] = [\text{rref} \mid E_k \cdots E_3 E_2 E_1] = [I \mid A^{-1}]. \]

\hfill \Box

\textbf{Example 2.4.1.} Find the inverse of the matrix

\[ A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 4 & 5 \\ 2 & 3 & -3 \end{pmatrix} \]

over \( \mathbb{R} \).
Construct the matrix \((A|I)\) and use row operations to bring the left side to the form of \(I_3\).

\[
\begin{pmatrix}
1 & 2 & -1 & 1 & 0 & 0 \\
0 & 4 & 5 & 0 & 1 & 0 \\
2 & 3 & -3 & 0 & 0 & 1
\end{pmatrix}
\]

Add two times the first row to the third row:

\[
\begin{pmatrix}
1 & 2 & -1 & 1 & 0 & 0 \\
0 & 4 & 5 & 0 & 1 & 0 \\
0 & -1 & -1 & -2 & 0 & 1
\end{pmatrix}
\]

Multiply the third row by \(-1\), and switch the second and third rows:

\[
\begin{pmatrix}
1 & 2 & -1 & 1 & 0 & 0 \\
0 & 1 & 1 & 2 & 0 & -1 \\
0 & 4 & 5 & 0 & 1 & 0
\end{pmatrix}
\]

Subtract four times the second row from the third:

\[
\begin{pmatrix}
1 & 2 & -1 & 1 & 0 & 0 \\
0 & 1 & 1 & 2 & 0 & -1 \\
0 & 0 & 1 & -8 & 1 & 4
\end{pmatrix}
\]

Subtract two times the second row from the first:

\[
\begin{pmatrix}
1 & 0 & -3 & -3 & 0 & 2 \\
0 & 1 & 1 & 2 & 0 & -1 \\
0 & 0 & 1 & -8 & 1 & 4
\end{pmatrix}
\]

Add three times the third row the the first, and subtract the third row from the second:

\[
\begin{pmatrix}
1 & 0 & 0 & -27 & 3 & 14 \\
0 & 1 & 0 & 10 & -1 & -5 \\
0 & 0 & 1 & -8 & 1 & 4
\end{pmatrix}
\]

The inverse of \(A\) is now on the right side:

\[
A^{-1} = \begin{pmatrix}
-27 & 3 & 14 \\
10 & -1 & -5 \\
-8 & 1 & 4
\end{pmatrix}
\]
Example 2.4.2. Show that the matrix

\[
B = \begin{pmatrix}
1 & 0 & 2 \\
0 & 2 & 1 \\
2 & 0 & 1
\end{pmatrix}
\]

is singular in \( \mathbb{Z}_3 \).

Construct the matrix \((B|I)\) and use row operations to bring the left side to the form of \(I_3\).

\[
\begin{pmatrix}
1 & 0 & 2 & 1 & 0 & 0 \\
0 & 2 & 1 & 0 & 1 & 0 \\
2 & 0 & 1 & 0 & 0 & 1
\end{pmatrix},
\]

subtracting 2 times the first row from the third row (modulo 3):

\[
\begin{pmatrix}
1 & 0 & 2 & 1 & 0 & 0 \\
0 & 2 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1
\end{pmatrix}.
\]

The reduced row-echelon form of the matrix is not \(I_3\), so \(B\) is singular in \(\mathbb{Z}_3\).

2.5 Diagonal, Triangular and Symmetric Matrices

Definition 2.5.1. A square matrix in which all the entries off the main diagonal are zero is called **diagonal matrix**.

Example 2.5.1. Some examples for diagonal matrices:

\[
\begin{pmatrix}
3 & 0 \\
0 & 7
\end{pmatrix}, \quad
\begin{pmatrix}
2 & 0 & 0 \\
0 & -4 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad
\begin{pmatrix}
-2 & 0 & 0 \\
0 & 6 & 0 \\
0 & 0 & 5
\end{pmatrix}.
\]

Definition 2.5.2. A square matrix in which the entries below the main diagonal are all zero is called **upper triangular**.

Example 2.5.2. Some examples for upper triangular matrices:

\[
\begin{pmatrix}
3 & 4 \\
0 & 7
\end{pmatrix}, \quad
\begin{pmatrix}
2 & 0 & 3 \\
0 & -4 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad
\begin{pmatrix}
2 & 7 & 3 \\
0 & -4 & 5 \\
0 & 0 & -3
\end{pmatrix}.
\]
Definition 2.5.3. A square matrix in which the entries above the main diagonal are all zero is called lower triangular.

Example 2.5.3. Some examples for lower triangular matrices:

\[
\begin{pmatrix}
3 & 0 \\
9 & 7
\end{pmatrix},
\begin{pmatrix}
2 & 0 & 0 \\
0 & -4 & 0 \\
6 & 5 & 0
\end{pmatrix},
\begin{pmatrix}
2 & 0 & 0 \\
5 & -4 & 0 \\
-3 & 0 & 0
\end{pmatrix}.
\]

Definition 2.5.4. A matrix that is either upper triangular or lower triangular is called triangular.

Example 2.5.4. Some examples for triangular matrices:

\[
\begin{pmatrix}
3 & 0 \\
9 & 7
\end{pmatrix},
\begin{pmatrix}
2 & 0 & 0 \\
0 & -4 & 0 \\
0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
2 & 7 & 0 \\
0 & -4 & 7 \\
0 & 0 & 6
\end{pmatrix}.
\]

Definition 2.5.5. A square matrix is called symmetric if \(A^T = A\).

Example 2.5.5. Some examples for symmetric matrices over \(\mathbb{R}\) and also over \(\mathbb{C}\):

\[
\begin{pmatrix}
2 & 7 & 0 \\
7 & -4 & 3 \\
0 & 3 & 6
\end{pmatrix},
\begin{pmatrix}
1 & 21 & 9 \\
21 & 11 & 0 \\
9 & 0 & 17
\end{pmatrix}.
\]

Example 2.5.6. Show that: If \(S\) is an invertible symmetric matrix, then \(S^{-1}\) is also symmetric.

We have to show that \(S^{-1}\) is also symmetric, that is \((S^{-1})^T = S^{-1}\). Is this true? We can use the rule for transpose and inverse: \((A^{-1})^T = (A^T)^{-1}\), and that the matrix \(S\) is symmetric, which means \(S^T = S\), so

\[(S^{-1})^T = (S^T)^{-1} = (S^{-1})^{-1},\]

which proves that \(S^{-1}\) is symmetric.

Definition 2.5.6. A square matrix is called skew-symmetric if \(A^T = -A\).
Example 2.5.7. 
\[
\begin{pmatrix}
0 & 7 & 0 \\
-7 & 0 & 3 \\
0 & -3 & 0
\end{pmatrix}
\quad
\begin{pmatrix}
0 & -21 & -9 \\
21 & 0 & 0 \\
9 & 0 & 0
\end{pmatrix}
\]
are skew-symmetric matrices over \( \mathbb{R} \) (and also over \( \mathbb{C} \)). Notice that the entries along the main diagonal must be zero.

Example 2.5.8. 
\[
\begin{pmatrix}
1 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix},
\begin{pmatrix}
1 & 1 & 1 \\
1 & 0 & 0 \\
1 & 0 & 1
\end{pmatrix}
\]
are skew-symmetric matrices in \( \mathbb{Z}_2 \). Notice that the entries along the main diagonal can be either 0 or 1, because \(-1 \equiv 1 \pmod{2}\).

Example 2.5.9. Show that: If \( A \) is an invertible skew-symmetric matrix, then \( A^{-1} \) is also skew-symmetric.

Definition 2.5.7. A square matrix \( A \) is called nilpotent if \( A^k = 0 \) for some positive integer \( k \).

Example 2.5.10. The matrices
\[
A = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}
\quad \text{and} \quad
B = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
7 & 0 & 0 & 0 & 0 \\
0 & 3 & 0 & 0 & 0 \\
0 & 0 & 4 & 0 & 0 \\
0 & 0 & 0 & -1 & 0
\end{pmatrix}
\]
are nilpotent over \( \mathbb{R} \), because \( A^3 = 0 \) and \( B^5 = 0 \).

Definition 2.5.8. A real square-matrix \( A \) is called orthogonal if \( A^T = A^{-1} \).

Example 2.5.11. Some examples for orthogonal matrices:
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix},
\begin{pmatrix}
\frac{1}{2} & -\frac{\sqrt{3}}{2} \\
\frac{\sqrt{3}}{2} & \frac{1}{2}
\end{pmatrix},
\begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & -1 \\
0 & 1 & 0
\end{pmatrix}.
\]