Appendix A
Supplementary Material

A.1 Cayley-Hamilton Theorem

Theorem A.1.1. *Every matrix is a root of its characteristic polynomial.*

Example A.1.1. Let $q(t) = (t-4)^4 - t^2 + 8t$. Evaluate $q(C)$, where

$$C = \begin{pmatrix} 5 & -1 \\ 1 & 3 \end{pmatrix}.$$

The characteristic equation of the matrix is $\lambda^2 - 8\lambda + 16$. Cayley-Hamilton Theorem says that the matrix C is a root of this polynomial, which means if we plug C in to the equation, then we get zero: $C^2 - 8C + 16I$. So we try to rearrange the given polynomial $q(t)$:

$$q(t) = (t-4)^4 - t^2 + 8t = \left[(t-4)^2\right]^2 - (t^2 - 8t + 16) + 16.$$

Using the Cayley-Hamilton theorem:

$$q(C) = \left[(C-4I)^2\right]^2 - (C^2 - 8C + 16I) + 16I = 16I = \begin{pmatrix} 16 & 0 \\ 0 & 16 \end{pmatrix}.$$

A.2 Exponential of Matrices

In differential equations you will use the exponent of a matrix, so in this worksheet we will have some questions with that. The exponent of a square matrix A is defined by the Taylor series:

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots + \frac{A^n}{n!} + \cdots$$

As you will learn the solution of the vector differential equation

$$\frac{d}{dt}\vec{y} = A\vec{y}$$

with initial condition $\vec{y}(0) = \vec{y}_0$ is

$$\vec{y}(t) = e^{At}\vec{y}_0,$$

where

$$e^{At} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \cdots + \frac{(At)^n}{n!} + \cdots$$

(a) If A is a diagonal matrix, then it is very easy to calculate e^A. For example, let

$$A = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}.$$

Then

$$\begin{aligned} e^A &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} \frac{(3)^2}{2!} & 0 \\ 0 & \frac{(2)^2}{2!} \end{pmatrix} + \begin{pmatrix} \frac{(3)^3}{3!} & 0 \\ 0 & \frac{(2)^3}{3!} \end{pmatrix} + \cdots \\ &= \begin{pmatrix} 1 + 3 + \frac{(3)^2}{2!} + \cdots & 0 \\ 0 & 1 + 2 + \frac{(2)^2}{2!} + \cdots \end{pmatrix} \\ &= \begin{pmatrix} e^3 & 0 \\ 0 & e^2 \end{pmatrix} \end{aligned}$$

Problem A.2.1. What is e^B, where $B = \begin{pmatrix} 7 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{pmatrix}$?

Solution.

$$e^B = \begin{pmatrix} e^7 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-3} \end{pmatrix}.$$

Watch the middle term: $e^0 = 1$.

(b) If the matrix A is diagonalizable (when A has as many distinct eigenvectors as its dimension), then $A = PDP^{-1}$, $A^2 = PD^2P^{-1}$, $\dots A^n = PD^nP^{-1}$, $\dots$, so

$$\begin{aligned} e^A &= I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots + \frac{A^n}{n!} + \cdots \\ &= I + PDP^{-1} + \frac{PD^2P^{-1}}{2!} + \frac{PD^3P^{-1}}{3!} + \cdots + \frac{PD^nP^{-1}}{n!} + \cdots \\ &= P\left(I + D + \frac{D^2}{2!} + \frac{D^3}{3!} + \cdots + \frac{D^n}{n!} + \cdots\right)P^{-1} \\ &= Pe^DP^{-1} \end{aligned}$$

Problem A.2.2. Let

$$A = \begin{pmatrix} 5 & -3 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

Find the eigenvalues of A. For each of its eigenvalue, find a basis for the corresponding eigenspace. Explain why the matrix A is diagonalizable, find its diagonal form D and the matrix P so that $A = PDP^{-1}$. Calculate e^A.

Solution. The eigenvalues of A are 5, 3, and 2. A basis for the eigenspace corresponding to $\lambda = 5$ is $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

A basis for the eigenspace corresponding to $\lambda = 3$ is $\begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}$.

A basis for the eigenspace corresponding to $\lambda = 2$ is $\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$.

In this case

$$P = \begin{pmatrix} 1 & 3 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{and} \quad D = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Note: Depending the basis vectors you chose, you could have different matrix for P.

$$e^A = Pe^D P^{-1} = \begin{pmatrix} 1 & 3 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^5 & 0 & 0 \\ 0 & e^3 & 0 \\ 0 & 0 & e^2 \end{pmatrix} \frac{1}{2} \begin{pmatrix} 2 & -3 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} e^5 & -\frac{3}{2}e^5 + \frac{3}{2}e^3 & -\frac{1}{2}e^5 + \frac{3}{2}e^3 - e^2 \\ 0 & e^3 & e^3 - e^2 \\ 0 & 0 & e^2 \end{pmatrix}$$

(c) Let

$$C = \begin{pmatrix} -2 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & -4 & -4 \end{pmatrix}.$$

Find the eigenvalue(s) of C. For each eigenvalue find a basis for the corresponding eigenspace. Explain why the matrix C is not diagonalizable.

In the case when the matrix C has only one eigenvalue λ, there is a "trick" to calculate its exponent e^C. Instead of C we write $C - \lambda I + \lambda I$ and we use the property of exponents ($e^{a+b} = e^a e^b$).

$$e^C = e^{C-\lambda I+\lambda I} = e^{C-\lambda I} e^{\lambda I} = e^{(C-\lambda I)} e^{\lambda I}.$$

In the last part λI is diagonal so we can easily calculate $e^{\lambda I}$ as described in part (a). The matrix $C - \lambda I$ is nilpotent (a consequence of the Cayley-Hamilton Theorem), so the infinite Taylor series will truncate to a finite sum.

For the matrix C given above, show that $(C - \lambda I)^2 \neq 0$ but $(C - \lambda I)^3 = 0$. So the Taylor series of $e^{C-\lambda I}$ truncates to the first three terms:

$$e^{C-\lambda I} = I + C - \lambda I + \frac{1}{2!}(C - \lambda I)^2.$$

Problem A.2.3. Calculate e^C (by calculating $e^{\lambda I}$ and $e^{C-\lambda I}$ and then using that $e^C = e^{C-\lambda I} e^{\lambda I}$).

Solution. The eigenvalue of C is $\lambda = -2$, and a basis for its eigenspace is

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

So C is not diagonalizable because there is only one basis vector for the eigenspace(s).

$$\begin{aligned}
e^C &= e^{(C+2I)}e^{-2I} \\
&= \left(I + C + 2I + \frac{1}{2!}(C+2I)^2\right) e^{-2}I \\
&= \left(\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} + \begin{pmatrix}0 & -1 & 0\\ 0 & 2 & 1\\ 0 & -4 & -2\end{pmatrix} + \frac{1}{2}\begin{pmatrix}0 & -2 & -1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\right) e^{-2} \\
&= e^{-2}\begin{pmatrix}1 & -2 & -\frac{1}{2}\\ 0 & 3 & 1\\ 0 & -4 & 1\end{pmatrix}.
\end{aligned}$$