

Appendix C

C.1 Rules of Matrix Arithmetic

Theorem C.1.1 (Properties of Matrix Arithmetic). *Assuming that the sizes of the matrices are such that the indicated operations can be performed, the following rules of matrix arithmetic are valid. Here A , B and C denote matrices, 0 is a zero matrix, a and b are scalars, and r and s are positive integers.*

1. $A + B = B + A$

Commutative law for addition

2. $A + (B + C) = (A + B) + C$

Associative law for addition

3. $a(B + C) = aB + aC = (B + C)a$

4. $a(B - C) = aB - aC = (B - C)a$

5. $(a + b)C = aC + bC = C(a + b)$

6. $(a - b)C = aC - bC = C(a - b)$

7. $a(bC) = (ab)C$

8. $A + 0 = 0 + A = A$

9. $A - A = 0$

10. $0 - A = -A$

11. $A(BC) = (AB)C$

Associative law for multiplication

12. $A(B + C) = AB + AC$

Left distributive law

13. $(B + C)A = BA + CA$

Right distributive law

14. $A(B - C) = AB - AC$

15. $(B - C)A = BA - CA$

16. $a(BC) = (aB)C = B(aC)$

17. $A0 = 0A = 0$

18. $AI = IA = A$

19. $A^r A^s = A^{r+s}$

20. $(A^r)^s = A^{rs}$

21. $A^0 = I$

22. $(A + B)^T = A^T + B^T$

23. $(A - B)^T = A^T - B^T$

24.* $(AB)^T = B^T A^T$

25. $(A^T)^T = A$

26. $(aB)^T = aB^T$

27.* $\text{tr}(A + B) = \text{tr}(A) + \text{tr}(B)$

28.* $\text{tr}(A - B) = \text{tr}(A) - \text{tr}(B)$

29.* $\text{tr}(aB) = a \cdot \text{tr}(B)$

30.* $\text{tr}(AB) = \text{tr}(BA)$

31. $(A^{-1})^{-1} = A$

$$32. (A^n)^{-1} = (A^{-1})^n$$

$$33. (aA)^{-1} = \frac{1}{a}A^{-1}, \text{ for any nonzero scalar } a.$$

$$34.* (AB)^{-1} = B^{-1}A^{-1}.$$

$$35. (A^T)^{-1} = (A^{-1})^T.$$

$$36.* \det(I) = 1.$$

$$37.* \det(AB) = \det(A) \det(B).$$

$$38.* \det(aA) = a^n \det(A), \text{ where } n \text{ is the size of the matrix } A.$$

$$39.* \det(A^{-1}) = \frac{1}{\det(A)}.$$

$$40.* A \operatorname{adj}(A) = I \det(A) \text{ and so } A^{-1} = \frac{\operatorname{adj}(A)}{\det(A)}.$$

C.2 Equivalent Statements

Theorem C.2.1. *If A is an $n \times n$ matrix, and if $T_A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a multiplication by A , then the following are equivalent.*

1. *A is invertible.*
2. *$A\vec{x} = \vec{0}$ has only the trivial solution.*
3. *The reduced row-echelon form of A is I_n .*
4. *$A\vec{x} = \vec{b}$ is consistent for every $n \times 1$ matrix \vec{b} .*
5. *$A\vec{x} = \vec{b}$ has exactly one solution for every $n \times 1$ matrix \vec{b} .*
6. *$\det(A) \neq 0$.*
7. *The range of T_A is \mathbb{R}^n .*
8. *T_A is one-to-one.*
9. *The column vectors of A are linearly independent.*
10. *The row vectors of A are linearly independent.*
11. *The column vectors of A span \mathbb{R}^n .*
12. *The row vectors of A span \mathbb{R}^n .*
13. *The column vectors of A form a basis for \mathbb{R}^n .*
14. *The row vectors of A form a basis for \mathbb{R}^n .*
15. *A has rank n .*
16. *A has nullity 0 .*
17. *$\lambda = 0$ is not an eigenvalue of A .*