Directions: You have 90 minutes to answer the following questions. You must show all your work as clearly as possible. You may use a calculator, but if a numerical answer is required, you must give its exact value. The last three pages contain a list of useful formulas. You may tear these pages out.

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(1) (Worksheet I) Fill in the blanks. (You do not have to explain and you do not have to evaluate any of the integrals.)

(a) If $R$ is the region bounded by $y = x^2$ and $y = 2x^2 - 4$, then \( \int_R f(x, y) \, dA \) as an iterated integral is

\[
\text{______________________________}
\]

(b) If $D$ is the lower left quadrant of the disk $(x - 1)^2 + (y - 1)^2 \leq 1$, then \( \int_D f(x, y) \, dA \) as an iterated integral is

\[
\text{______________________________}
\]

(c) An iterated integral equal to the volume of the region between the graph of $f(x, y) = x^2 + y^2$ and the plane $z = 3$ is

\[
\text{______________________________}
\]

(d) A lamina occupies the region of the $xy$-plane bounded by the curves $y = e^{2x}$ and $y = 3e^x - 2$. At each point $(x, y)$, its mass density is equal to the distance from that point to the origin.

An iterated integral equal to the total mass of the lamina is

\[
\text{______________________________}
\]
(2) (Worksheet I) Find the center of mass of the portion of the unit ball for which all the points have positive coordinates. Assume that the region has constant density.

(Hint: You may use spherical coordinates to evaluate the integrals.)
(3) (Worksheet II) Fill in the blanks. (You do not have to explain and you do not have to evaluate any of the integrals.)

(a) The iterated integral \( \int_0^4 \int_{x^2}^{16-(4-x)^2} f(x, y) \, dy \, dx \) with the reversed order of integration becomes

________________________

(b) The integral \( \int_0^3 \int_{|x|}^{\sqrt{18-x^2}} \sqrt{x^2 + y^2} \, dy \, dx \) has the following form in polar coordinates

________________________

(c) The integral \( \int_2^4 \int_{\pi/2}^{\pi} r^2 \cos(\theta) \, d\theta \, dr \) has the following form in cartesian coordinates

________________________

(d) The average distance to the origin for points in the solid bounded by the surfaces

\[ z = 8 - \sqrt{x^2 + y^2} \quad \text{and} \quad z = 0 \]

is

________________________
(4) (Problem 18, Page 778) Use cylindrical or spherical coordinates to evaluate the following integral:

\[
\int_{\sqrt{3}}^{\sqrt{3-x^2}} \int_{\sqrt{x^2-y^2}}^{1} \int_{1}^{z^2} \frac{1}{z^2} \, dz \, dy \, dx.
\]

(You must give the exact value of the integral.)
(5) Determine whether each of the following statements is True or False. (You do not have to explain.)

(a) If $B$ is the bottom half of the unit circle, then $\int_B dA \leq 0$.

(b) A bead is made by drilling a cylindrical hole of radius 1 mm through a sphere of radius 5 mm. A triple integral in cylindrical coordinates representing the volume of the bead is

$$\int_0^{2\pi} \int_0^5 \int_0^{\sqrt{25-r^2}} r \, dz \, dr \, d\theta.$$ 

(c) The iterated integrals

$$\int_{-1}^1 \int_0^1 \int_0^{1-x^2} f \, dz \, dy \, dx$$

and

$$\int_0^1 \int_0^1 \int_0^{\sqrt{1-z}} f \, dx \, dy \, dz$$

are equal.

(d) The region of integration of the triple iterated integral $\int_0^1 \int_1^2 \int_0^y f(x, y, z) \, dz \, dy \, dx$ lies above a square in the $xy$-plane and below a plane.
(6) (Problems 23, 24, Page 767) A sphere of radius 5, centered at the origin, is cut by the plane $z = 3$ into two parts. If $R$ denotes the smaller of the two parts, set up, but do not evaluate, an iterated triple integral representing the volume of $R$ using

(a) cartesian coordinates

(b) cylindrical coordinates

(c) spherical coordinates
Useful Formulas

• Cylindrical coordinates:
  \((r, \theta, z), \ 0 \leq r < \infty, \ 0 \leq \theta \leq 2\pi, \ -\infty < z < \infty.\)
  \(x = r \cos \theta, \ y = r \sin \theta, \ z = z\)
  \(dV = r \, dr \, d\theta \, dz\)

• Spherical Coordinates:
  \((\rho, \theta, \phi), \ 0 \leq \rho < \infty, \ 0 \leq \theta \leq 2\pi, \ 0 \leq \phi \leq \pi.\)
  \(x = \rho \cos \theta \sin \phi, \ y = \rho \sin \theta \sin \phi, \ z = \rho \cos \phi.\)
  \(dV = \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi.\)

• Center of Mass:

  If a two-dimensional object has density \(\delta(x, y)\) at the point \((x, y)\) and occupies a region \(R\), then the coordinates \((\bar{x}, \bar{y})\) of its center of mass are given by
  \[
  \bar{x} = \frac{1}{m} \int_R x \, \delta \, dA, \quad \bar{y} = \frac{1}{m} \int_R y \, \delta \, dA, 
  \]
  where \(m = \int_R \delta \, dA\) is the total mass of the body.

  If a three-dimensional object has density \(\delta(x, y, z)\) at the point \((x, y, z)\) and occupies a region \(W\), then the coordinates \((\bar{x}, \bar{y}, \bar{z})\) of its center of mass are given by
  \[
  \bar{x} = \frac{1}{m} \int_W x \, \delta \, dV, \quad \bar{y} = \frac{1}{m} \int_W y \, \delta \, dV, \quad \bar{z} = \frac{1}{m} \int_W z \, \delta \, dV, 
  \]
  where \(m = \int_W \delta \, dV\) is the total mass of the body.
Here $a, b, c, d$ are constants.

A Short Table of Indefinite Integrals

I. Basic Functions

1. $\int x^n \, dx = \frac{1}{n+1}x^{n+1} + C, \; (n \neq -1)$
2. $\int \frac{1}{x} \, dx = \ln |x| + C$
3. $\int a^x \, dx = \frac{1}{\ln a}a^x + C$
4. $\int \ln x \, dx = x \ln x - x + C$

5. $\int \sin ax \, dx = -\frac{1}{a} \cos ax + C$
6. $\int \cos ax \, dx = \frac{1}{a} \sin ax + C$
7. $\int \tan ax \, dx = -\frac{1}{a} \ln |\cos ax| + C$

II. Products of $e^x$, $\cos x$, and $\sin x$

8. $\int e^{ax} \sin(bx) \, dx = \frac{1}{a^2 + b^2}e^{ax}[a \sin(bx) - b \cos(bx)] + C$
9. $\int e^{ax} \cos(bx) \, dx = \frac{1}{a^2 + b^2}e^{ax}[a \cos(bx) + b \sin(bx)] + C$
10. $\int \sin(ax) \sin(bx) \, dx = \frac{1}{b^2 - a^2}[a \cos(ax) \sin(bx) - b \sin(ax) \cos(bx)] + C, \; a \neq b$
11. $\int \cos(ax) \cos(bx) \, dx = \frac{1}{b^2 - a^2}[b \cos(ax) \sin(bx) - a \sin(ax) \cos(bx)] + C, \; a \neq b$
12. $\int \sin(ax) \cos(bx) \, dx = \frac{1}{b^2 - a^2}[b \sin(ax) \sin(bx) + a \cos(ax) \cos(bx)] + C, \; a \neq b$

III. Product of Polynomial $p(x)$ with $\ln x, e^x, \cos x,$ and $\sin x$

13. $\int x^n \ln x \, dx = \frac{1}{n+1}x^{n+1} \ln x - \frac{1}{(n+1)^2}x^{n+1} + C, \; n \neq -1, x > 0$
14. $\int p(x)e^{ax} \, dx = \frac{1}{a}p(x)e^{ax} - \frac{1}{a^2}p'(x)e^{ax} + \frac{1}{a^3}p''(x)e^{ax} - \cdots + C$
   $(+ - + - + - + \ldots) \; \text{(signs alternate)}$
15. $\int p(x) \sin ax \, dx = -\frac{1}{a^2}p(x) \cos(ax) + \frac{1}{a^2}p'(x) \sin(ax) + \frac{1}{a^3}p''(x) \cos(ax) - \cdots + C$
   $(- + + - + + - + \ldots) \; \text{(signs alternate in pairs)}$
16. $\int p(x) \cos ax \, dx = \frac{1}{a}p(x) \sin(ax) + \frac{1}{a^2}p'(x) \cos(ax) - \frac{1}{a^3}p''(x) \sin(ax) - \cdots + C$
   $(+ + - + + - + \ldots) \; \text{(signs alternate in pairs)}$
IV. Integer Powers of \( \sin x \) and \( \cos x \)

17. \[ \int \sin^n x \, dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx, \quad n \text{ positive} \]

18. \[ \int \cos^n x \, dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x \, dx, \quad n \text{ positive} \]

19. \[ \frac{1}{\sin^m x} \, dx = -\frac{1}{m-1} \frac{\cos x}{\sin^{m-1} x} + \frac{m-2}{m-1} \int \frac{1}{\sin^{m-2} x} \, dx, \quad m \neq 1, m \text{ positive} \]

20. \[ \frac{1}{\sin x} \, dx = \frac{1}{2} \ln \left| \frac{\cos x - 1}{\cos x + 1} \right| + C \]

21. \[ \frac{1}{\cos^m x} \, dx = \frac{1}{m-1} \frac{\sin x}{\cos^{m-1} x} + \frac{m-2}{m-1} \int \frac{1}{\cos^{m-2} x} \, dx, \quad m \neq 1, m \text{ positive} \]

22. \[ \frac{1}{\cos x} \, dx = \frac{1}{2} \ln \left| \frac{\sin x + 1}{\sin x - 1} \right| + C \]

23. \[ \int \sin^m x \cos^n x \, dx : \]

If \( n \) is odd, let \( w = \sin x \).

If both \( m \) and \( n \) are even and non-negative, convert all to \( \sin x \) or all to \( \cos x \) (using \( \sin^2 x + \cos^2 x = 1 \)), and use IV-17 or IV-18.

If \( m \) and \( n \) are even and one of them is negative, convert to whichever function is in the denominator and use IV-19 or IV-21.

The case in which both \( m \) and \( n \) are even and negative is omitted.

V. Quadratic in the Denominator

24. \[ \int \frac{1}{x^2 + a^2} \, dx = \frac{1}{a} \arctan \left( \frac{x}{a} \right) + C, \quad a \neq 0 \]

25. \[ \int \frac{bx + c}{x^2 + a^2} \, dx = \frac{b}{2} \ln |x^2 + a^2| + \frac{c}{a} \arctan \left( \frac{x}{a} \right) + C, \quad a \neq 0 \]

26. \[ \int \frac{1}{(x-a)(x-b)} \, dx = \frac{1}{(a-b)} (\ln |x-a| - \ln |x-b|) + C, \quad a \neq b \]

27. \[ \int \frac{cx + d}{(x-a)(x-b)} \, dx = \frac{1}{(a-b)} [(ac + d) \ln |x-a| - (bc + d) \ln |x-b|] + C, \quad a \neq b \]

VI. Integrand involving \( \sqrt{a^2 + x^2}, \sqrt{a^2 - x^2}, \sqrt{x^2 - a^2}, a > 0 \)

28. \[ \int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \left( \frac{x}{a} \right) + C \]

29. \[ \int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln |x + \sqrt{x^2 \pm a^2}| + C \]

30. \[ \int \sqrt{a^2 + x^2} \, dx = \frac{1}{2} \left( x \sqrt{a^2 + x^2} + a^2 \int \frac{1}{\sqrt{a^2 + x^2}} \, dx \right) + C \]

31. \[ \int \sqrt{x^2 - a^2} \, dx = \frac{1}{2} \left( x \sqrt{x^2 - a^2} + a^2 \int \frac{1}{\sqrt{x^2 - a^2}} \, dx \right) + C \]