

FIRST ORDER DIFFERENTIAL EQUATIONS

Students frequently ask “how do I know which method to use?”. It’s a good question and one that isn’t necessarily easy to answer except with “practice!”. If you write down a random first order differential equation, there’s a good chance it will be either very hard or impossible to solve. However in MA2132 we only learn a few methods so it’s probable that an exam question will fall into one of those categories, maybe after doing some algebra.

(1) A Strategy for Solving First Order Equations:

- (a) First use algebra to put the equation in normal form $y' = f(x, y)$. In other words, solve for the derivative. Remember that y is the function and x is the variable but make the obvious changes if different letters are used for the function and variable.
- (b) Can you factor the right hand side of the equation as $f(x, y) = g(x)h(y)$? In this case the equation is **separable**, you can write it as

$$\int \frac{dy}{h(y)} = \int g(x)dx$$

and then integrate. Don’t forget $+C$!

- (c) Can you write the equation in the form $y' = a(x)y + f(x)$? If so, the equation is **linear** and you should multiply the equation by the integrating factor

$$u(x) = e^{-\int a(x)dx}$$

and then write the equation as $u(x)y' - u(x)a(x)y = u(x)f(x)$. The left hand side is now $(u(x)y)'$ by the product rule, so you can solve by integrating both sides. Don’t forget $+C$!

- (d) Can you write the equation in the form $y' = a(x)y + f(x)y^n$? If so, the equation is a **Bernoulli equation**, and you can turn it into a linear equation with the substitution $z = y^{1-n}$ and then some algebra. For example, if $n = 4$ then $z = y^{-3}$, so $y = z^{-1/3}$ and $y' = -\frac{1}{3}z^{-4/3}z'$ by the chain rule (remember the variable is x .) In this case, the linear equation simplifies to $z' = -3a(x)z - 3f(x)$. Don’t forget $+C$ when you solve the linear equation!
- (e) If none of the above methods work do some algebra to rewrite the differential equation in the form $Mdx + Ndy = 0$, treating $y' = dy/dx$ as a fraction. Partial derivatives follow the same rules as ordinary derivatives except we treat the other letter as a constant.

The equation $Mdx + Ndy = 0$ is **exact** provided

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

and if this is the case, then we solve the equations

$$\frac{\partial F}{\partial x} = M, \quad \frac{\partial F}{\partial y} = N$$

to find $F(x, y)$ by integrating M with respect to x . Remember that since we are doing “partial” integration, the “constant” of integration is $\phi(y)$. Then take the partial derivative of F with respect to y , and compare it to N to solve for the constant of integration $\phi(y)$. It makes no difference if you start with the equation $\frac{\partial F}{\partial y} = N$ instead, but remember you have to switch the role of x and y . Don’t forget that the final answer is $F(x, y) = C$.

Examples:

- (a) $y' = e^{x-y}$. This equation is separable because $e^{x-y} = e^x e^{-y}$
- (b) $y' = 1 + x + y + xy$. This equation is separable because $1 + x + y + xy = (1+x)(1+y)$. It is also linear because $1 + x + y + xy = (1+x)y + (1+x)$, so $a(x) = 1+x$ and $f(x) = 1+x$.
- (c) $y' = y + y^3$. This equation is separable, and it is also a Bernoulli equation with $n = 3$.
- (d) $y' = xy + y^3$. This is a Bernoulli equation but not a separable equation.
- (e) $xy' - \sin(x)/y = \ln(x)y$. The normal form of this equation is

$$y' = \frac{\ln(x)}{x}y + \frac{\sin(x)}{x}y^{-1}$$

so it is a Bernoulli equation with $n = -1$.

- (f) $y' = (\cos(y) - ye^x)/(e^x + x \sin(y))$. There is no way to factor the right hand side of this equation so it is not separable. Linear and Bernoulli equations can never have terms of the form $\sin(y)$ where y is the *function*. Rewrite the equation as

$$(-\cos(y) + ye^x)dx + (e^x + x \sin(y))dy = 0$$

and set $M = -\cos(y) + ye^x$ and $N = e^x + x \sin(y)$. The equation is exact because

$$\frac{\partial M}{\partial y} = \sin(y) + e^x = \frac{\partial N}{\partial x}$$

(2) Autonomous Equations:

Autonomous equations have the form $y' = f(y)$ so they are a special kind of separable equation, and the examples we are interested in can usually be solved using partial fractions. But generally we are more interested in studying the general solution graphically, we can sketch the direction field very easily because the derivative only depends on y and not on the variable t .

First find the equilibrium (constant) solutions which correspond to the roots of $f(y)$. Then sketch $f(y)$ and from this draw the phase line to determine where $y' = f(y)$ is positive and negative and thus where y is increasing and decreasing, and also to see whether the equilibrium points are stable or unstable.

(3) The Logistic Equation:

The logistic equation has the form $P' = r_0(1 - P/K)P$ where r_0 is the *natural* reproductive rate, $r_0(1 - P/K)$ is the reproductive rate, and K is the carrying capacity. The equation is both separable and Bernoulli, but is probably easier to solve as a separable equation (using partial fractions) which gives the implicit solution

$$\frac{P}{K - P} = \frac{P_0}{K - P_0} e^{r_0 t}$$

where P_0 is the initial population. Generally you will see “word” problems which can be solved by algebra using the implicit solution. The carrying capacity K may be described as “the population stabilizes at” or “the maximum possible population is” but you will need to use common sense to work out exactly what it means in a particular question.