An \( n \)-dimensional first order linear system is a list of \( n \) first order, linear, differential equations, in \( n \) unknown functions of one variable. It may be written as the matrix/vector equation

\[
x' = Ax + f
\]

where \( x \) is an \( n \)-dimensional vector containing the unknown functions and \( A \) is the \( n \times n \) coefficient matrix. For example, the 3-dimensional linear system

\[
\begin{align*}
x'_1 &= -x_1 + 5x_2 + 3x_3 + e^t \\
x'_2 &= x_2 + x_3 \\
x'_3 &= -2x_2 - 2x_3 - t^2
\end{align*}
\]

has

\[
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
-1 & 5 & 3 \\
0 & 1 & 1 \\
0 & -2 & -2
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
e^t \\
0 \\
-t^2
\end{pmatrix}
\]

Note that the book uses \( y \)'s instead of \( x \)'s in some sections, so you should be comfortable using either letter. We will almost always use \( t \) for the variable.

(1) **Homogeneous Solution:** Solve the homogeneous equation

\[
x' = Ax
\]

to find \( n \) linearly independent vector solutions \( x_1, x_2, \ldots, x_n \), and hence the homogeneous solution \( x_h = c_1x_1 + c_2x_2 + \cdots + c_nx_n \). You can check whether \( x_1, x_2, \ldots, x_n \) are linearly independent by calculating the Wronskian

\[
W(t_0) = \det \begin{pmatrix}
x_1(t_0) & x_2(t_0) & \cdots & x_n(t_0)
\end{pmatrix}
\]

at any point \( t_0 \) in the domain of \( x_1, x_2, \ldots, x_n \), and showing that it is non-zero. Note that the Wronskian is formed by first placing the column vector solutions next to each other to form a square matrix, substituting \( t_0 \) and then taking the determinant of the matrix.

To find the \( n \) linearly independent vector solutions, you first need to find the eigenvalues \( \lambda \) of \( A \), by solving the characteristic equation \( \det(A - \lambda I) = 0 \) for \( \lambda \), and then find the corresponding eigenvectors \( v \), by solving the vector equation \( (A - \lambda I)v = 0 \) for \( v \). If there are repeated eigenvalues you may need to also find generalized eigenvectors - this case will be explained below.

(a) **Distinct Real Eigenvalues:** If \( \lambda \) is a real eigenvalue which appears once as a root of the characteristic equation, and \( v \) is the corresponding eigenvector then \( x = e^{\lambda t}v \) is a vector solution of the homogeneous equation.

(b) **Distinct Complex Eigenvalues:** If \( \lambda = \alpha + i\beta \) is a complex eigenvalue then so is its complex conjugate \( \lambda = \alpha - i\beta \). Each of these eigenvalues will provide one vector solution, but since the eigenvalues are closely related, it is probably easier to use \( \lambda = \alpha + i\beta \) to find both solutions. Find the corresponding eigenvector \( v \) and split it using vector addition as \( v = v_1 + iv_2 \), where \( v_1 \) and \( v_2 \) have real entries. Then the two vector solutions are

\[
x_1 = e^{\alpha t}(\cos(\beta t)v_1 - \sin(\beta t)v_2)
\]

and

\[
x_2 = e^{\alpha t}(\sin(\beta t)v_1 + \cos(\beta t)v_2)
\]

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Example:

\[
\begin{pmatrix}
1 & 1 \\
-1 & 1 \\
\end{pmatrix}
\]

The eigenvalues are \(1 + i\), and \(1 - i\). Use the eigenvalue \(1 + i\) to find the eigenvector by solving

\[
\begin{pmatrix}
-i & 1 \\
-1 & -i \\
\end{pmatrix} \mathbf{v} = \mathbf{0}
\]

Then

\[
\mathbf{v} = \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \mathbf{v}_1 + i\mathbf{v}_2
\]

and so the two corresponding vector solutions are

\[
x_1 = e^{t \mathbf{v}_1} = e^{t \begin{pmatrix} 1 \\ 0 \end{pmatrix}} - \sin(t) e^{t \begin{pmatrix} 0 \\ 1 \end{pmatrix}}
\]

and

\[
x_2 = e^{t \mathbf{v}_2} = e^{t \begin{pmatrix} 1 \\ 0 \end{pmatrix}} + \cos(t) e^{t \begin{pmatrix} 0 \\ 1 \end{pmatrix}}
\]

(c) **Repeated Eigenvalues:** If an eigenvalue is repeated we need to analyse the matrix \(A\) more carefully to find the corresponding vector solutions.

**Definition 1.** The Algebraic Multiplicity (AM) of an eigenvalue \(\lambda\) is the number of times it appears as a root of the characteristic equation \(\det(A - \lambda I) = 0\).

For each eigenvalue \(\lambda\), we need to find the same number of vector solutions as its Algebraic Multiplicity. If \(AM = 1\), then we are in one of the situations described above so the more difficult (and interesting) case is when \(AM > 1\).

**Definition 2.** The Geometric Multiplicity (GM) of an eigenvalue \(\lambda\) is the number of linearly independent eigenvectors corresponding to \(\lambda\) (or the dimension of the eigenspace). An equivalent (and probably more helpful) definition is that the Geometric Multiplicity is the number of degrees of freedom in the eigenvector equation \((A - \lambda I)\mathbf{v} = \mathbf{0}\).

These two numbers always have the property that \(1 \leq GM \leq AM\). In the case where \(GM = AM\), we have “enough” eigenvectors to produce solutions of the form \(\mathbf{x} = e^{t \mathbf{v}_1}\) corresponding to \(\lambda\) and we proceed as above. Thus the most interesting case is when \(GM < AM\). To see what happens in general, you are referred to the textbook, however to avoid being cast adrift in a sea of confusing abstraction and notation we shall only consider a special case here.

**Definition 3.** If \(\lambda\) is an eigenvalue for \(A\), then \(\mathbf{v}\) is a corresponding **generalized eigenvector** if \((A - \lambda I)^d \mathbf{v} = \mathbf{0}\) for some positive integer \(d\).

Note that an eigenvector is also a generalized eigenvector (take \(d = 1\)) but as \(d\) increases there will be more solutions which are not eigenvectors.

Consider a real eigenvalue \(\lambda\) for which \(AM = 2\) and \(GM = 1\), so that we are looking for two vector solutions corresponding to \(\lambda\) but we only have one linearly independent eigenvector \(\mathbf{v}_1\), and hence only one vector solution \(\mathbf{x}_1 = e^{t \mathbf{v}_1}\).

Solve the equation \((A - \lambda I)^2 \mathbf{v} = \mathbf{0}\) which will have two degrees of freedom, so there will be two linearly independent generalized eigenvectors. It is strongly recommended (though not necessary) that you choose the eigenvector \(\mathbf{v}_1\) as one of the generalized...
eigenvectors, since this gives us the solution \( x_1 = e^{\lambda t}v_1 \) which we already know. If \( v_2 \) is the other generalized eigenvector then, the second solution corresponding to \( \lambda \) is
\[
    x_2 = e^{\lambda t}(I + t(A - \lambda I))v_2.
\]

For more details on how this solution arises, see the Appendix on matrix exponentials.

(2) **Particular Solution:** To find the particular solution \( x_p \) for an inhomogeneous first order linear system
\[
ex' = Ax + f
\]
we use the Variation of Parameters method.

**Definition 4. The Fundamental Matrix**
\[
Y = (x_1 \ x_2 \ \cdots \ x_n)
\]
is the \( n \times n \) matrix formed by writing the \( n \) column vector solutions of the homogeneous equation next to each other.

It is easy to check that \( Y' = AY \) and that \( Y \) is invertible because its determinant is the Wronskian and hence non-zero. It can then be shown that the particular vector solution is
\[
x_p = Y \int Y^{-1}f \ dt
\]
You should compute \( x_p \) in the following order:
(a) Find \( Y^{-1} \) which is an \( n \times n \) matrix.
(b) Find \( Y^{-1}f \) which is a vector.
(c) Find \( \int Y^{-1}f \ dt \) which is a vector. You compute the integral of a vector by finding an anti-derivative of each entry, and this is a case where you can choose the constants of integration to be 0 because we are only looking for one particular solution.
(d) Find \( x_p = Y \int Y^{-1}f \ dt \) which is a vector.

(3) **General Solution:** The general solution is the vector
\[
x = x_h + x_p = c_1x_1 + c_2x_2 + \cdots + c_n x_n + x_p
\]

(4) **Initial Value Problems:** An IVP will have the form
\[
ex' = Ax + f, \ x(t_0) = x_0
\]
In this case you can solve for the constants \( c_1, c_2, \ldots, c_n \). Substitute the initial condition into the general solution, and this will give you \( n \) equations in the \( n \) unknown constants.
Examples: See if you can verify the solutions to the following problems.

(1) In this example we have a repeated eigenvalue but there are sufficient eigenvectors

\[
x' = \begin{pmatrix} 2 & 2 & -4 \\ 2 & -1 & -2 \\ 4 & 2 & -6 \end{pmatrix} x
\]

The characteristic equation for the matrix is

\[-(\lambda + 1)(\lambda + 2)^2 = 0\]

so the eigenvalues are \(-1\) and \(-2\). First consider the eigenvalue \(\lambda = -2\), which has Algebraic Multiplicity 2, and for which the eigenvector equation is

\[
\begin{pmatrix} 4 & 2 & -4 \\ 2 & 1 & -2 \\ 4 & 2 & -4 \end{pmatrix} v = 0.
\]

This equation has two degrees of freedom so the eigenvalue \(\lambda = 2\) has Geometric Multiplicity 2 and thus it has two eigenvectors. For example we could choose

\[
v_1 = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}
\]

The eigenvalue \(\lambda = -1\) has Algebraic Multiplicity 1 and a corresponding eigenvector

\[
v_3 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}.
\]

The general solution of the system is

\[
x = c_1 e^{-2t} \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c_3 e^{-t} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}
\]

(2) In this example there is a repeated eigenvalue and we need to find a generalized eigenvector.

\[
x' = \begin{pmatrix} -1 & 2 & 1 \\ 0 & -1 & 0 \\ -1 & -3 & -1 \end{pmatrix} x
\]

The characteristic equation for the matrix is (as in the previous example)

\[-(\lambda + 1)(\lambda + 2)^2 = 0\]

and so the eigenvalues are \(-1\) and \(-2\). First consider the eigenvalue \(\lambda = -2\), which has Algebraic Multiplicity 2, and for which the eigenvector equation is

\[
\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ -1 & -3 & -1 \end{pmatrix} v = 0
\]

which only has one degree of freedom. We can choose the eigenvector

\[
v_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}
\]
but now we need to also find a generalized eigenvector. Compute

\[(A - \lambda I)^2 = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ -1 & -3 & -1 \end{pmatrix}^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 0 \end{pmatrix}\]

and then solve the equation

\[
 \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 0 \end{pmatrix}v = 0
\]

which has two degrees of freedom. As always, we should choose the eigenvector \(v_1\) as one solution. The other solution \(v_2\) is a generalized eigenvector and must be linearly independent with \(v_1\), so for example we can choose

\[v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\]

The two vector solutions of the system are then

\[x_1 = e^{-2t} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}\]

and

\[x_2 = e^{-2t}(I + t \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ -1 & -3 & -1 \end{pmatrix}) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = e^{-2t} \begin{pmatrix} 1 + t \\ 0 \\ -t \end{pmatrix}\]

The eigenvalue \(\lambda = -1\) has Algebraic Multiplicity 1 and a corresponding eigenvector

\[v_3 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}\]

and so the general solution of the system is

\[x = c_1e^{-2t} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_2e^{-2t} \begin{pmatrix} 1 + t \\ 0 \\ -t \end{pmatrix} + c_3e^{-t} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}\]

(3) Solve the initial value problem for following system of equations

\[x' = -x + y + t, y' = x - y - t, \ x(0) = 1, y(0) = 1\]

to find formulas for \(x(t)\) and \(y(t)\).

First put the system of equations into matrix/vector form \(x' = Ax + f\) where

\[A = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad f = \begin{pmatrix} t \\ -t \end{pmatrix}\]

and then find the eigenvalues of \(A\) which are 0 and -2. The corresponding eigenvectors are

\[v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}\]

so a Fundamental Matrix for the homogeneous system is

\[Y = \begin{pmatrix} 1 & -e^{-2t} \\ 1 & e^{-2t} \end{pmatrix}\]
Now we compute the particular solution $x_p$

(a) $Y^{-1} = \frac{1}{2} \begin{pmatrix} 1 & -e^{2t} \\ -e^{2t} & e^{2t} \end{pmatrix}$

(b) $Y^{-1}f = \begin{pmatrix} 0 \\ -e^{2t} \end{pmatrix}$

(c) $\int Y^{-1}f \, dt = \begin{pmatrix} 0 \\ -\frac{t}{2} + \frac{1}{4} \end{pmatrix}$

(d) $x_p = Y \int Y^{-1}f \, dt = \begin{pmatrix} \frac{t}{2} - \frac{1}{4} \\ -\frac{t}{2} + \frac{1}{4} \end{pmatrix}$

The general solution is

$$x = x_h + x_p = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} \frac{t}{2} - \frac{1}{4} \\ -\frac{t}{2} + \frac{1}{4} \end{pmatrix}$$

Finally, remember that $x = \begin{pmatrix} x \\ y \end{pmatrix}$ and substitute the initial values to find $c_1 = 1$ and $c_2 = -1/4$. The solutions are

$$x = \frac{3}{4} + \frac{1}{4} e^{-2t} + \frac{t}{2}$$

$$y = \frac{5}{4} - \frac{1}{4} e^{-2t} - \frac{t}{2}$$

**APPENDIX: MATRIX EXPONENTIALS**

**Definition 5.** Let $A$ be a square matrix and let $t$ be a variable then

$$e^{tA} = I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots$$

This is the definition but we NEVER (well almost never) use it to actually find $e^{tA}$ because computing an infinite series of matrices is too hard. Usually we want to compute $e^{tA}v$ since this is a solution of $x' = Ax$ for any vector $v$.

Let $\lambda$ be an eigenvalue of $A$, and note that $tA = t(\lambda I + A - \lambda I)$. Then, because $t\lambda I$ is a diagonal matrix, we can use the usual law of exponents and so

$$e^{tA} = e^{t\lambda I} e^{t(A - \lambda I)} = e^{t\lambda} (I + t(A - \lambda I) + \frac{t^2}{2!} (A - \lambda I)^2 + \frac{t^3}{3!} (A - \lambda I)^3 + \cdots)$$

Now if $v_1$ is an eigenvector, then $(A - \lambda I)v_1 = 0$, so

$$e^{tA}v_1 = e^{t\lambda}Iv_1 = e^{t\lambda}v_1$$

and if $v_2$ is a generalized eigenvector, then $(A - \lambda I)^d v_2 = 0$, so

$$e^{tA}v_2 = e^{t\lambda} (I + t(A - \lambda I) + \frac{t^2}{2!} (A - \lambda I)^2 + \frac{t^3}{3!} (A - \lambda I)^3 + \cdots + \frac{t^{d-1}}{(d-1)!} (A - \lambda I)^{d-1})v_2$$

Finally, if you ever need to compute a matrix exponential, find a Fundamental Matrix $Y$ for the equation $x' = Ax$ and then use the formula

$$e^{tA} = Y \cdot Y(0)^{-1}$$