Solutions of Second Order Linear Equations

A second order linear equation has the normal form $y'' + p(t)y' + q(t)y = f(t)$ where p, q, and f are functions of the independent variable t. If it comes in the form $A(t)y''+B(t)y'+C(t)y = F(t)$ you should first divide through by $A(t)$ to put it in normal form. To find the solution you should follow the general procedure for linear problems.

(1) **Homogeneous Solution:** Solve the homogeneous equation

$$y'' + p(t)y' + q(t)y = 0$$

to find two linearly independent solutions y_1 and y_2, and hence the homogeneous solution $y_h = c_1y_1 + c_2y_2$. You can check whether y_1 and y_2 are linearly independent by calculating the Wronksian

$$W(t_0) = \det \begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix}$$

at any point t_0 in the domain of y_1 and y_2, and showing it is non-zero. Usually we choose t_0 to be 0 or 1 to make the calculation easier. However, even homogeneous, linear equations are hard to solve so we usually impose some restrictions.

(a) **Reduction of Order:** If you know one solution y_1, you can find the other one by substituting $y_2 = vy_1$ in the homogenous equation. Remember that v is a function of t when you compute derivatives. This will reduce the equation to one of the form $v'' + a(t)v' = 0$ which is really a first order, separable equation. To see this explicitly, make the substitution $u = v', u' = v''$, then solve for u. Then v is an anti-derivative of u (since we are looking for one function, you can choose the constant of integration to be 0) and you can find $y_2 = vy_1$ by substituting v.

(b) **Constant Coefficients:** If the equation has the form $y''+py'+qy = 0$ then substitute $y = e^{\lambda t}$ to obtain the characteristic equation $\lambda^2 + p\lambda + q = 0$. Find the roots by factoring or using the quadratic formula. There are three cases:

1. If the roots λ_1, λ_2 are real and distinct then $y_1 = e^{\lambda_1 t}$ and $y_2 = e^{\lambda_2 t}$.

2. If the roots are complex conjugates $\lambda_1 = a+ib$ and $\lambda_2 = a-ib$, then $y_1 = e^{at}\cos(bt)$ and $y_2 = e^{at}\sin(bt)$.

3. If there is a repeated real root λ then the solutions are $y_1 = e^{\lambda t}$ and $y_2 = te^{\lambda t}$. You can use reduction of order to see why y_2 has that form.

In all three cases y_1 and y_2 are linearly independent but you should still know how to check this using the Wronskian.

(c) **Cauchy-Euler Equations:** In this case the homogeneous equation has the form $t^2y'' + pty' + qy = 0$ (where p, q are constants) or equivalently

$$y'' + \frac{p}{t}y + \frac{q}{t^2}y = 0$$

provided we assume that $t > 0$. Make the substitution $y = t^r$ to obtain the equation $r(r-1)+pr+q = 0$, and solve it to find the values of r. Again there are three cases:

1. If r_1 and r_2 are real, distinct roots then $y_1 = t^{r_1}$ and $y_2 = t^{r_2}$.

2. If r is a repeated root then $y_1 = t^r$ and $y_2 = \ln(t)t^r$. Again, you can use reduction of order to verify the form of y_2.

3. We will not see the case where the roots are complex in this course. As an exercise in complex arithmetic see if you can work out what the solutions should be.

(2) **Particular Solution:** There are two ways to find the particular solution y_p. The Method of Undetermined Coefficients is easier but only works in some cases and there are also special cases to remember. Variation of Parameters works in all cases but is harder because it involves integration.

(a) **Undetermined Coefficients:** This method basically only works for equations of the form $y'' + py' + qy = f(t)$ where p and q are constants and $f(t)$ is a linear combination of products of polynomial, sin and cosine functions, and exponential functions. For example, it works when $f(t) = 3e^{-3t}\sin(2t) + 21t^2e^t\cos(3t)$ but not when $f(t) = 1/\cos(t)$.
The idea is to use the form of $f(t)$ to make an educated guess for y_p and then substitute in the differential equation and solve for the undetermined co-efficients. The precise form of y_p will be explained in class, and is also in the text book.
The special case occurs when a term you want to include in y_p is already present in y_h. In this case you must multiply the *bad* terms in y_p by powers of t until there are no repeats.
Example: $y'' - 2y' = t^2 + \sin(2t)$. The solutions of the homogeneous equation are $y_1 = 1$ and $y_2 = e^{2t}$. So you would choose

$$y_p = (At^2 + Bt + C \cdot 1)t + D\sin(2t) + E\cos(2t)$$

since 1 is already contained in y_h.

(b) **Variation of Parameters:** In this method $y_p = v_1y_1 + v_2y_2$ where y_1 and y_2 are solutions of the homogeneous equation. To find v_1 and v_2 you need to solve the Variation of Parameters (VOP) equations

$$\begin{aligned} v_1'y_1 + v_2'y_2 &= 0 \\ v_1'y_1' + v_2'y_2' &= f(t). \end{aligned}$$

You can find v_1' and v_2' by setting $v_1' = -v_2'y_2/y_1$ from the first equation and then substituting in the second. Integrate to find v_1 and v_2, and substitute back to get $y_p = v_1y_1 + v_2y_2$. This is another case where we are just looking for ONE function y_p so you can choose your constant of integration to be 0 when you compute v_1 and v_2.

(3) **General Solution:** $y = y_h + y_p = c_1y_1 + c_2y_2 + y_p$

(4) **Initial Value Problems:** An IVP will have the form

$$y'' + p(t)y' + q(t)y = f(t),\ y(t_0) = a, y'(t_0) = b.$$

In this case you can solve for the constants c_1 and c_2. First compute y' by differentiating the general solution, and then subsitute using the initial values. Solve the two resulting equations to find c_1 and c_2 and finally remember to write out your solution.

Examples: See if you can verify the solutions to the following problems.

(1) $y'' - 3y' + 2y = 2\sin(2t) - 3e^t,\ y(0) = 2, y'(0) = 3.$

The solution of the homogeneous equation is

$$y_h = c_1e^t + c_2e^{2t}.$$

To find the particular solution use undetermined coefficients and set

$$y_p = A\sin(2t) + B\cos(2t) + Cte^t.$$

Substitute in the original equation and compare coefficients to find $A = -1/10, B = 3/10, C = 3$ and so the general solution is

$$y = c_1e^t + c_2e^{2t} - \frac{1}{10}\sin(2t) + \frac{3}{10}\cos(2t) + 3te^t,$$

and the derivative of y is

$$y' = c_1e^t + 2c_2e^{2t} - \frac{1}{5}\cos(2t) - \frac{3}{5}\sin(2t) + 3(e^t + te^t).$$

Substituting the initial values into these formulas gives

$$\begin{aligned} 2 &= c_1 + c_2 + 3/10 \\ 3 &= c_1 + 2c_2 - 1/5 + 3 \end{aligned}$$

and so $c_1 = 16/5$ and $c_2 = -3/2$. The solution of the IVP is thus

$$y = \frac{16}{5}e^t - \frac{3}{2}e^{2t} - \frac{1}{10}\sin(2t) + \frac{3}{10}\cos(2t) + 3te^t.$$

(2) $t^2y'' + ty' - y = t^2,$ for $t > 0,\ y(1) = 1, y'(1) = 2$

The solution of the homogeneous equation is

$$y_h = c_1t + c_2\frac{1}{t}.$$

To find the particular solution use variation of parameters, so in this case

$$y_p = v_1t + v_2\frac{1}{t}$$

Now solve the two VOP equations with $y_1 = t, y_2 = 1/t, y_1' = 1, y_2' = -1/t^2$ and $f(t) = 1$ (why?) and you should find that $v_1' = 1/2$ and $v_2' = -t^2/2$. Integrating gives $v_1 = t/2$ and $v_2 = -t^3/6$, so

$$y_p = \frac{t}{2}t + \frac{-t^3}{6}\frac{1}{t} = \frac{t^2}{3},$$

and the general solution is

$$y = c_1t + c_2\frac{1}{t} + \frac{t^2}{3}.$$

Solving for the constants with the initial conditions gives $c_1 = 1$ and $c_2 = -1/3$ so the solution of the IVP is

$$y = t - \frac{1}{3t} + \frac{t^2}{3}.$$