Polytechnic University

MA 2132 | Final | March 10, 2003

Print Name:

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Directions: You have **two hours** to answer the following questions. You must show all your work as neatly and clearly as possible and indicate the final answer clearly. You may use a calculator **but you must show your work for integrals and derivatives.**. There are formulas on the last page which you may detach.

Problem	Possible	Points
1	15	
2	20	
3	20	
4	15	
5	20	
6	10	
Total	100	

(1) (15 points) Solve the IVP

$$x' = \frac{x}{t} + \frac{t}{x}, \qquad x(1) = 1,$$

and find the interval of existence for the solution.

(2) (20 points) Consider the IVP

$$y' = y + y^2, \qquad y(0) = 1.$$

(a) Use Euler's method with a stepsize of $h = 0.1$ to find the approximate value of $y(0.2)$.

(b) Use Improved Euler's method (also called the RK2 method) with stepsize of $h = 0.2$ to find the approximate value of $y(0.2)$.

(c) Find the exact solution of the IVP and determine which of your solutions from the first two parts of the question is more accurate.

(3) (20 points) Solve the IVP for the system of first order equations, to find formulas for $x_1(t)$ and $x_2(t)$

$$x_1' = x_2 + e^t, \quad x_2' = x_1 + e^{-t}, \qquad x_1(0) = 3, \quad x_2(0) = -2.$$

(4) (15 points) Consider the fourth order ODE

$$y^{(4)} - 2y^{(3)} + 3y'' - 2y' + 2y = 0.$$

(a) Given that $y_1 = e^t \sin(t)$ is a solution of the ODE, find three other solutions y_2, y_3 and y_4 and verify that the four solutions are linearly independent using the Wronskian.

(b) Find a matrix A so that the equation can be written as a first order system of equations $\mathbf{x}' = A\mathbf{x}$. What are the eigenvalues of A?

(5) (20 points) Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & 3 \\ 0 & -1 & -2 \end{pmatrix}.$$

(a) Show that the eigenvalues of A are -1 and 1, and that one of them is a repeated eigenvalue.

(b) Find the algebraic and geometric multiplicity of each eigenvalue.

Continuation of Problem 5.

(c) For the matrix $A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & 3 \\ 0 & -1 & -2 \end{pmatrix}$, solve the IVP

$$\mathbf{x}' = A\mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}.$$

Hint: Use your solutions to (a) and (b).

(6) (10 points) State whether each of the following statements is TRUE or FALSE. You need not show your work.

(a) If $\mathbf{x_1}$ and $\mathbf{x_2}$ are solutions of the first order linear system $\mathbf{x}' = A(t)\mathbf{x}$ then $4\mathbf{x_1} - \mathbf{x_2}$ is also a solution.

(b) If $\mathbf{x_1}$ and $\mathbf{x_2}$ are solutions of the first order linear system $\mathbf{x}' = A(t)\mathbf{x} + \mathbf{f}(t)$ then $4\mathbf{x_1} - \mathbf{x_2}$ is also a solution.

(c) Every initial value problem of the form

$$y' = f(x)g(y), \quad y(0) = 0,$$

has a unique solution.

(d) If $y = \cos(x)$ is a solution of a second order differential equation then $y = \sin(x)$ is also a solution.

(e) If A is an $n \times n$ matrix, λ is an eigenvalue for A and $\mathbf{v}$ is a generalized eigenvector for λ, then $e^{tA}\mathbf{v} = e^{\lambda t}\mathbf{v}$.

Formula Sheet

(1) **Integration By Parts:** $\int u(x)v'(x)\,dx = u(x)v(x) - \int u'(x)v(x)\,dx$

(2) **Partial Fractions Integral:** If $c \neq d$ then

$$\int \frac{ax+b}{(x-c)(x-d)}\,dx = \frac{1}{c-d}((ac+b)\ln|x-c| - (ad+b)\ln|x-d|) + K$$

(3) **The Logistic Equation:** $P' = r_0(1 - P/K)P$ has the implicit general solution

$$\frac{P}{K-P} = \frac{P_0}{K-P_0}e^{r_0 t}.$$

(4) **Variation of Parameters for Second Order Equations:** If y_1 and y_2 are linearly independent solutions of the equation $y'' + p(t)y' + q(t)y = 0$, then $y_p = v_1y_1 + v_2y_2$ is a particular solution of the equation $y'' + p(t)y' + q(t)y = f(t)$, where v_1 and v_2 satisfy the VOP equations

$$\begin{aligned} v_1'y_1 + v_2'y_2 &= 0 \\ v_1'y_1' + v_2'y_2' &= f(t). \end{aligned}$$

(5) **Matrix Exponential:** If A is a square matrix and t is a variable then

$$e^{tA} = I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots$$

(6) **Generalized Eigenvector:** If λ is an eigenvalue for a square matrix A, then $\mathbf{v}$ is a corresponding generalized eigenvector if $(A - \lambda I)^d\mathbf{v} = \mathbf{0}$ for some positive integer d.

(7) **Variation of Parameters for First Order Linear Systems:** If $\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n}$ are linearly independent vector solutions of the n-dimensional homogeneous linear system $\mathbf{x}' = A\mathbf{x}$, then

$$\mathbf{x_p} = Y\int Y^{-1}\mathbf{f}\,dt$$

is a particular solution of the system $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$ where Y is the $n \times n$ matrix

$$\begin{bmatrix}\mathbf{x_1} & \mathbf{x_2} & \ldots & \mathbf{x_n}\end{bmatrix}.$$