Polytechnic University

MA 2132 — Final — May 9th, 2003

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Directions: You have **two hours** to answer the following questions. You must show all your work as neatly and clearly as possible and indicate the final answer clearly. You may use a calculator **but you must show your work for integrals and derivatives.**. There are formulas on the last page which you may detach.

Problem	Possible	Points
1	15	
2	20	
3	20	
4	15	
5	20	
6	10	
Total	100	

(1) (15 points) Consider the differential equation

$$y^{(3)} - y = y'' - y'$$

(a) Find an equivalent system of first order equations, and write it as a matrix equation $\mathbf{x}' = A\mathbf{x}$.

(b) What are the eigenvalues of A?

(2) (20 points) Consider the autonomous first order differential equation

$$y' = (y+1)^2(y-1)$$

(a) Find the equilibrium solutions of the equation and determine whether they are (asymptotically) stable or unstable.

(b) Sketch a direction field for the equation in the region $-2 \le t \le 2, -2 \le y \le 2$. Clearly indicate the equilibrium solutions, and sketch the particular solution for which $y(0) = 0$.

(c) Use the Euler method with a stepsize $h = 0.1$ to find the approximate value of $y(0.2)$ for the IVP

$$y' = (y+1)^2(y-1), \quad y(0) = 0.$$

(3) (20 points) Solve the IVP for the following first order linear system.

$$\mathbf{x}' = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} t \\ t^2 \end{bmatrix}, \quad \mathbf{x}(1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(4) (15 points) Consider the IVP for following first order linear system where x and y are functions of the variable t.

$$x' = y, \quad y' = -x, \quad x(0) = -1, \quad y(0) = 0$$

(a) Verify that the solution of the system is $x = -\cos(t), y = \sin(t)$.

(b) Sketch the solution in the (x, y)-plane. Indicate carefully the initial value, and the direction of the graph as the parameter t increases.

(5) (20 points - continued on next page) Consider the following linear system of first order equations, where x_1, x_2 and x_3 are functions of the variable t:

$$\begin{aligned} x_1' &= 4x_1 + 2x_2 \\ x_2' &= -3x_1 - x_2 \\ x_3' &= 3x_1 + 2x_2 + x_3 \end{aligned}$$

(a) Write the system as a matrix equation $\mathbf{x}' = A\mathbf{x}$, and show that the eigenvalues of A are 1 and 2, where one of them is a repeated eigenvalue.

(b) Find the algebraic and geometric multiplicity of each eigenvalue of A.

(c) Find the general solution of the system, and write your answer as formulas for $x_1(t)$, $x_2(t)$ and $x_3(t)$.

(6) (10 points) Suppose that $y_1 = e^t \cos(t), y_2 = t$ and $y_3 = 2t - 7$ are solutions of a linear, homogeneous differential equation with constant coefficients. State whether each the following statements is TRUE or FALSE. You do not need to show your work.

(a) 1 must also be a solution of the equation.

(b) $2t^2 - 7t$ must also be a solution of the equation.

(c) $e^t \sin(t) + t$ must also be a solution of the equation.

(d) $\cos(t)$ must also be a solution of the equation.

(e) The degree of the equation is at least 4.

Formula Sheet

(1) **Integration By Parts:** $\displaystyle\int u(x)v'(x)\,dx = u(x)v(x) - \int u'(x)v(x)\,dx$

(2) **Partial Fractions Integral:** If $c \neq d$ then

$$\int \frac{ax+b}{(x-c)(x-d)}dx = \frac{1}{c-d}((ac+b)\ln|x-c| - (ad+b)\ln|x-d|) + K$$

(3) **The Logistic Equation:** $P' = r_0(1-P/K)P$ has the implicit general solution

$$\frac{P}{K-P} = \frac{P_0}{K-P_0}e^{r_0 t}$$

(4) **Variation of Parameters for Second Order Equations:** If y_1 and y_2 are linearly independent solutions of the equation $y'' + p(t)y' + q(t)y = 0$, then $y_p = v_1y_1 + v_2y_2$ is a particular solution of the equation $y'' + p(t)y' + q(t)y = f(t)$, where v_1 and v_2 satisfy the VOP equations

$$\begin{aligned} v_1'y_1 + v_2'y_2 &= 0 \\ v_1'y_1' + v_2'y_2' &= f(t). \end{aligned}$$

(5) **Matrix Exponential:** If A is a square matrix and t is a variable then

$$e^{tA} = I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots$$

(6) **Generalized Eigenvector:** If λ is a an eigenvalue for a square matrix A, then $\mathbf{v}$ is a corresponding generalized eigenvector if $(A-\lambda I)^d\mathbf{v} = \mathbf{0}$ for some positive integer d.

(7) **Variation of Parameters for First Order Linear Systems:** If $\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n}$ are linearly independent vector solutions of the n-dimensional homogeneous linear system $\mathbf{x}' = A\mathbf{x}$, then

$$\mathbf{x_p} = Y\int Y^{-1}\mathbf{f}dt$$

is a particular solution of the system $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$ where Y is the $n \times n$ matrix

$$\begin{bmatrix}\mathbf{x_1} & \mathbf{x_2} & \ldots & \mathbf{x_n}\end{bmatrix}$$