Polytechnic University

MA 2132 — Final — March 6, 2006

Print Name:
Signature:
ID #:
Instructor/Section:

Directions: You have **two hours** to answer the following questions. You must show all your work as neatly and clearly as possible and indicate the final answer clearly. You may use a calculator **but you must show your work for integrals and derivatives**. There are formulas on the last two pages which you may detach.

If you are feeling ill you should inform the proctor. The proctor will note your name, Poly ID and accept any written statement(s) that you may wish to make regarding your illness.

Problem	Possible	Points
1	15	
2	18	
3	20	
4	15	
5	20	
6	12	
Total	100	

YOUR SIGNATURE: ______________________________

(1) Consider the initial value problem

$$xy' + 2y = \pi \sin(\pi x^2), \quad y(-1) = \frac{5}{2}.$$

(a) Solve the initial value problem. Give the solution in explicit form.

(b) Find the interval of existence of the solution.

(2) A rabbit population, $P(t)$, (in thousands) is modelled by the logistic equation

$$P' = \frac{1}{15}\left(1 - \frac{1}{80}P\right)P$$

where the units of t are months, and the initial population is 60 thousand.

(a) Use the Improved Euler Method (also called the RK2 method) with a stepsize of $h = 2$ to find the approximate size of the population after four months. Round your numbers to 4 decimal places.

(b) Find the **exact** size of the population after four months.

YOUR SIGNATURE: ______________________________________

(3) Solve the following IVP for the system of first order equations, to find formulas for $x(t)$ and $y(t)$.

$$x' = 4x - y + 16e^{3t}, \quad y' = 5x - 2y, \quad x(0) = 0, \quad y(0) = 4$$

(4) Assume that $y_1(t) = 3t^2$ and $y_2(t) = 1 + e^t \sin(3t)$ are solutions of a homogeneous, linear differential equation with constant real coefficients.

(a) What can you say about the order of the differential equation?

(b) List a set of fundamental solutions of such a differential equation.

(c) *(10 points)* Find an example for such a differential equation. Show your work.

(5) (Continued on the next page) Consider the matrix

$$A = \begin{pmatrix} 5 & 0 & 3 \\ 0 & 3 & 1 \\ 0 & 6 & 2 \end{pmatrix}$$

(a) *(3 points)* Show that the eigenvalues of A are 5 and 0, and that one of them is a repeated eigenvalue.

(b) *(4 points)* Find the algebraic and geometric multiplicity of each eigenvalue.

(c) *(13 points)* (Continued from the previous page) Find the general solution of the system $\frac{d}{dt}\vec{x} = A\vec{x}$, where

$$A = \begin{pmatrix} 5 & 0 & 3 \\ 0 & 3 & 1 \\ 0 & 6 & 2 \end{pmatrix}.$$

(6) (a) *(4 points)* Consider the first order system of equations

$$x' = y - 2, \quad y' = -x + 1, \quad x(0) = 1, \quad y(0) = 1$$

where x and y are both functions of t.
Verify by substitution that the solution of this system is

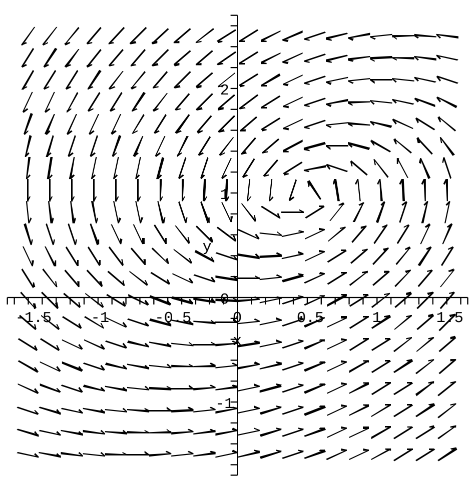
$$x = -\sin(t) + 1, \quad y = -\cos(t) + 2.$$

(b) *(8 points)* Consider the system

$$x' = -y + 1$$
$$y' = 1.02x - 0.5y.$$

(i) Find the (exact) equilibrium solution of this system: $(x, y) =$____________

(ii) The phase plane of this system is shown in the figure below for $-1.5 \leq x \leq 1.5$, $-1.5 \leq y \leq 2.5$. On this picture sketch the solution curve for the system with initial value $x(0) = 0.5$, $y(0) = -1$.



(iii) Describe in a full English sentence what happens with the solution as $t \to \infty$.

YOUR SIGNATURE:

FORMULA SHEET

(1) **Integration By Parts:** $\int u(x)v'(x)dx = u(x)v(x) - \int u'(x)v(x)dx$.

(2) **Partial Fractions Integral:** If $c \neq d$ then

$$\int \frac{ax+b}{(x-c)(x-d)}dx = \frac{1}{c-d}\Big((ac+b)\ln|x-c| - (ad+b)\ln|x-d|\Big) + K.$$

(3) **The Logistic Equation:** $P' = r_0\big(1 - P/K\big)P$ has the implicit general solution

$$\frac{P}{K-P} = \frac{P_0}{K-P_0}e^{r_0 t}.$$

(4) **For linear homogeneous d.e. with constant coefficients:** $y'' + by' + cy = 0$.

- If $b^2 - 4c > 0$, then r_1 and r_2 are two distinct solutions of the characteristic equation and
$$y = C_1 e^{r_1 t} + C_2 e^{r_2 t},$$
where C_1 and C_2 are constants.
- If $b^2 - 4c = 0$, then there is only one solution of the characteristic equation, $r = -b/2$, and
$$y = C_1 t e^{rt} + C_2 e^{rt}.$$
- If $b^2 - 4c < 0$, then the solutions of the characteristic equation are of the form $r = \alpha \pm \beta i$ and
$$y = C_1 e^{\alpha t}\cos(\beta t) + C_2 e^{\alpha t}\sin(\beta t).$$

(5) **For linear non-homogeneous d.e. with constant coefficients:**

If $f(x)$ is	then try $y_p(x)$ in the form of
polynomial $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$	polynomial of same degree $A_n x^n + A_{n-1}x^{n-1} + \cdots + A_0$
be^{kx}	Be^{kx}
$b\sin(ax)$ or $b\cos(ax)$	$B\sin(ax) + C\cos(ax)$

(6) **Variation of Parameters for Second Order Equations:** If y_1 and y_2 are linearly independent solutions of the equation $y'' + p(t)y' + q(t)y = 0$, then $y_p = v_1 y_1 + v_2 y_2$ is a particular solution of the equation $y'' + p(t)y' + q(t)y = f(t)$, where v_1 and v_2 satisfy the VOP equations

$$\begin{aligned} v_1' y_1 + v_2' y_2 &= 0 \\ v_1' y_1' + v_2' y_2' &= f(t). \end{aligned}$$

(7) **Matrix Exponential:** If A is a square matrix and t is a variable then

$$e^{tA} = I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots$$
$$e^{tA} = Y(t)Y^{-1}(0)$$

(8) **First Order Linear Systems with Complex Eigenvalues:** If $\lambda = \alpha + \beta i$ is an eigenvalue of A for the system of equations, $\vec{x}\,' = A\vec{x}$, with corresponding eigenvector $\vec{v} = \vec{p} + i\vec{q}$, then two solutions of the system are:

$$\vec{x}_1 = e^{\alpha t}[\cos(\beta t)\vec{p} - \sin(\beta t)\vec{q}]$$
$$\vec{x}_2 = e^{\alpha t}[\sin(\beta t)\vec{p} + \cos(\beta t)\vec{q}]$$

(9) **Generalized Eigenvector:** If λ is an eigenvalue for a square matrix A, then $\vec{v}$ is a corresponding generalized eigenvector if $(A - \lambda I)^d\vec{v} = \vec{0}$ for some positive integer d.

(10) **Variation of Parameters for First Order Linear Systems:** If $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$ are linearly independent vector solutions of the n-dimensional homogeneous linear system $\vec{x}\,' = A\vec{x}$, then

$$\vec{x}_p = Y \int Y^{-1}\vec{f}dt$$

is a particular solution of the system $\vec{x}\,' = A\vec{x} + \vec{f}$ where Y is the $n \times n$ matrix

$$\begin{bmatrix}\vec{x}_1 & \vec{x}_2 & \ldots & \vec{x}_n\end{bmatrix}.$$