Polytechnic University

MA 2132 — Final (Make-up) — May 8, 2006

Print Name:
Signature:
ID #:
Instructor/Section:

Directions: You have **two hours** to answer the following questions. You must show all your work as neatly and clearly as possible and indicate the final answer clearly. You may use a calculator **but you must show your work for integrals and derivatives.**. There are formulas on the last page which you may detach.

If you are feeling ill you should inform the proctor. The proctor will note your name, Poly ID and accept any written statement(s) that you may wish to make regarding your illness.

Problem	Possible	Points
1	20	
2	15	
3	20	
4	15	
5	20	
6	10	
Total	100	

YOUR SIGNATURE:

(1) Solve the IVP

$$x' = \frac{x}{t} - \frac{t}{x}, \quad x(1) = 1,$$

and find the interval of existence for the solution.

YOUR SIGNATURE:

(2) Consider the following differential equation

$$y' = 4x^3y - y.$$

(a) Suppose $y(1) = -3$. Use the Improved Euler's Method (also called the RK2 method) with stepsize $h = 0.1$ to find the approximate value of $y(1.2)$.

(b) Find the explicit solution of the IVP and evaluate $y(1.2)$.

YOUR SIGNATURE: ______

(3) Solve the IVP for the system of first order equations

$$x_1' = x_2 + \sin(t), \qquad x_2' = -x_1 + \cos(t), \qquad x_1(0) = 1, \quad x_2(0) = 2$$

to find formulas for $x_1(t)$ and $x_2(t)$.

(4) Find an example of a linear, homogeneous differential equation with constant real coefficients for which $y_1 = e^{-x}\sin(x)$ and $y_2 = x^2$ are solutions.
(Hint: What are the roots of the characteristic equation?)

YOUR SIGNATURE:

(5) (Continued on next page) Consider the matrix

$$A = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 1 & 0 \end{pmatrix}.$$

(a) Show that the eigenvalues of A are -1 and 2, and that one of them is a repeated eigenvalue.

(b) Find the algebraic and geometric multiplicity of each eigenvalue.

(c) Solve the IVP

$$\vec{x}\,' = A\vec{x}, \quad \vec{x}(0) = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

YOUR SIGNATURE:

(6) State whether each of the following statements is TRUE or FALSE. You do not need to show your work.

(a) If $\vec{x}_1$ and $\vec{x}_2$ are solutions of the first order linear system $\vec{x}\,' = A(t)\vec{x}$, then $2\vec{x}_1 + 3\vec{x}_2$ is also a solution.

(b) If $\vec{x}_1$ and $\vec{x}_2$ are solutions of the first order linear system $\vec{x}\,' = A(t)\vec{x} + \vec{f}(t)$, then $2\vec{x}_1 + 3\vec{x}_1$ is also a solution.

(c) Every initial value problem of the form $y' = f(y)g(x)$, $y(0) = 0$ has a unique solution.

(d) If $y = e^x$ is a solution of a second order differential equation, then $y = 2e^x$ is also a solution.

(e) If A is an $n \times n$ matrix, such that λ is an eigenvalue for A and a corresponding eigenvector is $\vec{v}$, then $e^{tA}\vec{v} = e^{\lambda t}\vec{v}$.

YOUR SIGNATURE:

FORMULA SHEET

(1) **Integration By Parts:** $\int u(x)v'(x)dx = u(x)v(x) - \int u'(x)v(x)dx$.

(2) **Partial Fractions Integral:** If $c \neq d$ then

$$\int \frac{ax+b}{(x-c)(x-d)}dx = \frac{1}{c-d}\big((ac+b)\ln|x-c| - (ad+b)\ln|x-d|\big) + K.$$

(3) **The Logistic Equation:** $P' = r_0\big(1 - P/K\big)P$ has the implicit general solution

$$\frac{P}{K-P} = \frac{P_0}{K-P_0}e^{r_0 t}.$$

(4) **For linear homogeneous d.e. with constant coefficients:** $y'' + by' + cy = 0$.

- If $b^2 - 4c > 0$, then r_1 and r_2 are two distinct solutions of the characteristic equation and
$$y = C_1e^{r_1t} + C_2e^{r_2t},$$
where C_1 and C_2 are constants.
- If $b^2 - 4c = 0$, then there is only one solution of the characteristic equation, $r = -b/2$, and
$$y = C_1te^{rt} + C_2e^{rt}.$$
- If $b^2 - 4c < 0$, then the solutions of the characteristic equation are of the form $r = \alpha \pm \beta i$ and
$$y = C_1e^{\alpha t}\cos(\beta t) + C_2e^{\alpha t}\sin(\beta t).$$

(5) **For linear non-homogeneous d.e. with constant coefficients:**

If $f(x)$ is	then try $y_p(x)$ in the form of
polynomial $a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0$	polynomial of same degree $A_nx^n + A_{n-1}x^{n-1} + \cdots + A_0$
be^{kx}	Be^{kx}
$b\sin(ax)$ or $b\cos(ax)$	$B\sin(ax) + C\cos(ax)$

(6) **Variation of Parameters for Second Order Equations:** If y_1 and y_2 are linearly independent solutions of the equation $y'' + p(t)y' + q(t)y = 0$, then $y_p = v_1y_1 + v_2y_2$ is a particular solution of the equation $y'' + p(t)y' + q(t)y = f(t)$, where v_1 and v_2 satisfy the VOP equations

$$\begin{aligned} v_1'y_1 + v_2'y_2 &= 0 \\ v_1'y_1' + v_2'y_2' &= f(t). \end{aligned}$$

(7) **Matrix Exponential:** If A is a square matrix and t is a variable then

$$e^{tA} = I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots$$
$$e^{tA} = Y(t)Y^{-1}(0)$$

(8) **First Order Linear Systems with Complex Eigenvalues:** If $\lambda = \alpha + \beta i$ is an eigenvalue of A for the system of equations, $\vec{x}\,' = A\vec{x}$, with corresponding eigenvector $\vec{v} = \vec{p} + i\vec{q}$, then two solutions of the system are:

$$\vec{x}_1 = e^{\alpha t}[\cos(\beta t)\vec{p} - \sin(\beta t)\vec{q}]$$
$$\vec{x}_2 = e^{\alpha t}[\sin(\beta t)\vec{p} + \cos(\beta t)\vec{q}]$$

(9) **Generalized Eigenvector:** If λ is an eigenvalue for a square matrix A, then $\vec{v}$ is a corresponding generalized eigenvector if $(A - \lambda I)^d \vec{v} = \vec{0}$ for some positive integer d.

(10) **Variation of Parameters for First Order Linear Systems:** If $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$ are linearly independent vector solutions of the n-dimensional homogeneous linear system $\vec{x}\,' = A\vec{x}$, then

$$\vec{x}_p = Y \int Y^{-1}\vec{f}dt$$

is a particular solution of the system $\vec{x}\,' = A\vec{x} + \vec{f}$ where Y is the $n \times n$ matrix

$$\left[\vec{x}_1 \; \vec{x}_2 \; \ldots \; \vec{x}_n\right].$$