Polytechnic University

MA 2132 MIDTERM FEBRUARY 18, 2003

Print Name:

Signature:

ID #:

Instructor/Section: Cornick Zauderer

Directions: You have **90 minutes** to answer the following questions. You must show all your work as neatly and clearly as possible and indicate the final answer clearly. You may use a calculator, **but you must show your work for integrals and derivatives**. There are formulas on the last page of the exam which you may detach.

Problem	Possible	Points
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

(1) (20 points) Consider the initial value problem

$$P' = e^{P-2t}, \qquad P(1) = 1$$

(a) Find an **explicit** formula for $P(t)$.

(b) What is the interval of existence for this solution?

(2) (20 points) In this problem we consider the autonomous ODE of the form $y' = f(y)$, where $f(y) = -y^3 - 3y^2 + 4y + 12$.

(a) Sketch a graph of $f(y)$ and use it to develop a **phase line** for the ODE which exhibits the **equilibrium points** and classifies them as being (asymptotically) **stable** or **unstable**.

(b) If $y(-2) = 0$, then $\lim_{t\to\infty} y(t) =$ _____.

(c) If $y(0) = -3$, then $\lim_{t\to\infty} y(t) =$ _____.

(d) If $y(0) = -2$, then $y(-3) =$ _____.

(3) (20 points) Solve the initial value problem for the second order ODE

$$y'' - 2y' = 1 + e^x + e^{2x}, \qquad y(0) = 1, \quad y'(0) = -1$$

(4) (20 points) Consider the homogenous, second order linear ODE

$$x^2y'' + 3xy' + y = 0, \text{ for } x > 0$$

(a) Verify that $y_1 = \dfrac{1}{x}$ is a solution.

(b) Use the substitution $y = vy_1$ to show that $y_2 = \dfrac{\ln(x)}{x}$ is another solution and then use the Wronskian to verify that y_1 and y_2 are linearly independent.

(c) Solve the IVP

$$x^2y'' + 3xy' + y = 0, \qquad y(1) = e, \quad y'(1) = \frac{1}{e}$$

(5) (20 points) A bacteria population $P(t)$ grows according to the logistic equation

$$P' = r_0(1 - \frac{P}{K})P$$

Suppose the initial population is 20 percent of the carrying capacity, and the population doubles after one hour.

(a) What is the *natural* reproductive rate for this population?

(b) At what time does the population reach 80 percent of its carrying capacity?

Formula Sheet

(1) **Integration By Parts:** $\int u(x)v'(x)\,\mathrm{dx} = u(x)v(x) - \int u'(x)v(x)\,\mathrm{dx}$

(2) **Partial Fractions Integral:** If $c \neq d$ then

$$\int \frac{ax+b}{(x-c)(x-d)}\,\mathrm{dx} = \frac{1}{c-d}((ac+b)\ln|x-c| - (ad+b)\ln|x-d|) + K$$

(3) **The Logistic Equation:** $P' = r_0(1-P/K)P$ has the implicit general solution

$$\frac{P}{K-P} = \frac{P_0}{K-P_0}e^{r_0 t}$$

(4) **Variation of Parameters:** If y_1 and y_2 are linearly independent solutions of the equation $y'' + p(t)y' + q(t)y = 0$, then $y_p = v_1y_1 + v_2y_2$ is a particular solution of the equation $y'' + p(t)y' + q(t)y = f(t)$, where v_1 and v_2 satisfy the VOP equations

$$\begin{aligned} v_1'y_1 + v_2'y_2 &= 0 \\ v_1'y_1' + v_2'y_2' &= f(t). \end{aligned}$$