Polytechnic University

MA 2132            MIDTERM            APRIL 7, 2006

Print Name:
Signature:
ID #:
Instructor/Section:

**Directions:** You have 90 minutes to answer the following questions. You must show all your work as neatly and clearly as possible and indicate the final answer clearly. You may use a calculator.

If you are feeling ill you should inform the proctor. The proctor will note your name, Poly ID and accept any written statement(s) that you may wish to make regarding your illness.

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<th>Problem</th>
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In each of the following parts, circle the **ALL** correct answers which describe the type of the given first order differential equation. You do not need to explain.

(a) $$(x^2 + 4)y' + 6x = 3xy$$
   (i) Separable.
   (ii) Autonomous.
   (iii) Linear.
   (iv) Bernoulli, but not linear.
   (v) None of the above are correct.

(b) $$y \cos(x)y' = e^x(y^2 + 1)$$
   (i) Separable.
   (ii) Autonomous.
   (iii) Linear.
   (iv) Bernoulli, but not linear.
   (v) None of the above are correct.

(c) $$y' = 2xy + 3e^x y^2$$
   (i) Separable.
   (ii) Autonomous.
   (iii) Linear.
   (iv) Bernoulli, but not linear.
   (v) None of the above are correct.

(d) $$\frac{dx}{dt} = kx(x - M)$$ (Here $k$ and $M$ are constants.)
   (i) Separable.
   (ii) Autonomous.
   (iii) Linear.
   (iv) Bernoulli, but not linear.
   (v) None of the above are correct.
(2) For each of the following parts (a)- (c) below, circle the ONE alternative that best completes the sentence. You do not need to explain.

(a) The equilibrium solution \( y = -2 \) of the differential equation \( \frac{dy}{dt} = y^3 - 4y \) is
   (i) asymptotically stable.
   (ii) unstable.

(b) If \( y(t) \) is the solution of the initial value problem \( y' = y^3 - 4y, \ y(2) = 1 \), then
   (i) \( \lim_{t \to \infty} y(t) = -\infty \).
   (ii) \( \lim_{t \to \infty} y(t) = -2 \).
   (iii) \( \lim_{t \to \infty} y(t) = 0 \).
   (iv) \( \lim_{t \to \infty} y(t) = 2 \).
   (v) \( \lim_{t \to \infty} y(t) = \infty \).

(c) If \( y(t) \) is the solution of the initial value problem \( y' = y^3 - 4y, \ y(1) = 2 \), then
   (i) \( \lim_{t \to \infty} y(t) = -\infty \).
   (ii) \( \lim_{t \to \infty} y(t) = -2 \).
   (iii) \( \lim_{t \to \infty} y(t) = 0 \).
   (iv) \( \lim_{t \to \infty} y(t) = 2 \).
   (v) \( \lim_{t \to \infty} y(t) = \infty \).
(3) Consider the initial value problem
\[ \cos(t) \frac{dy}{dt} - y \sin(t) = \frac{1}{t^2}, \quad y(\pi) = -\frac{2}{\pi}. \]

(a) Find an explicit formula for \( y(t) \).

(b) Find the interval of existence for this solution.
(4) Consider the differential equation

\[(1 + ye^{xy})dx + (2y + xe^{xy})dy = 0.\]

(a) Show that the differential equation is exact.

(b) Find the implicit solution of the differential equation.
(5) (a) Use the substitution \( y = t^r \) to find two solutions \( y_1 \) and \( y_2 \) of the homogeneous equation
\[
t^2 y'' + ty' - y = 0, \quad t > 0.
\]

(b) Use the Wronksian to verify that \( y_1 \) and \( y_2 \) are linearly independent.
(c) (Continued from the previous page) Use the variation of parameters method to find the general solution of

\[ t^2 y'' + ty' - y = \frac{1}{t^2}, \quad t > 0 \]
(6) Use the method of undetermined coefficients to find a general solution of

\[ \frac{d^2y}{dx^2} - 3 \frac{dy}{dx} + 2y = 2x + 1 + e^{3x}. \]
(7) A bacterium population \( P(t) \), grows according to the logistic differential equation

\[
\frac{dP}{dt} = r_0 \left( 1 - \frac{P}{2000} \right) P.
\]

Suppose the initial population is 25% of the carrying capacity, and the population doubles after 2 hours. Find the number of bacteria present after 20 hours.
(1) Integration By Parts: \[ \int u(x)v'(x)dx = u(x)v(x) - \int u'(x)v(x)dx. \]

(2) Partial Fractions Integral: If \( c \neq d \) then
\[
\int \frac{ax + b}{(x - c)(x - d)} \, dx = \frac{1}{c - d} \left[ (ac + b) \ln |x - c| - (ad + b) \ln |x - d| \right] + K.
\]

(3) The Logistic Equation: \( P' = r_0(1 - P/K)P \) has the implicit general solution
\[
\frac{P}{K - P} = \frac{P_0}{K - P_0} e^{r_0 t}.
\]

(4) For linear homogeneous d.e. with constant coefficients: \( y'' + by' + cy = 0 \).
- If \( b^2 - 4c > 0 \), then \( r_1 \) and \( r_2 \) are two distinct solutions of the characteristic equation and
\[
y = C_1 e^{r_1 t} + C_2 e^{r_2 t},
\]
where \( C_1 \) and \( C_2 \) are constants.
- If \( b^2 - 4c = 0 \), then there is only one solution of the characteristic equation, \( r = -b/2 \), and
\[
y = C_1 t e^{r t} + C_2 e^{r t}.
\]
- If \( b^2 - 4c < 0 \), then the solutions of the characteristic equation are of the form \( r = \alpha \pm \beta i \) and
\[
y = C_1 e^{\alpha t} \cos(\beta t) + C_2 e^{\alpha t} \sin(\beta t).
\]

(5) For linear non-homogeneous d.e. with constant coefficients:

| If \( f(x) \) is \( a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 \) polynomial \( b e^{kx} \) \( b \sin(ax) \) or \( b \cos(ax) \) \( b e^{kx} \) \( B e^{kx} \) \( B \sin(ax) + C \cos(ax) \) | then try \( y_p(x) \) in the form of \( A_n x^n + A_{n-1} x^{n-1} + \cdots + A_0 \) polynomial of same degree |

(6) Variation of Parameters: If \( y_1 \) and \( y_2 \) are linearly independent solutions of the equation \( y'' + p(t)y' + q(t)y = 0 \), then \( y_p = v_1 y_1 + v_2 y_2 \) is a particular solution of the equation \( y'' + p(t)y' + q(t)y = f(t) \), where \( v_1 \) and \( v_2 \) satisfy the VOP equations
\[
\begin{align*}
v_1' y_1 + v_2' y_2 &= 0 \\
v_1' y_1' + v_2' y_2' &= f(t).
\end{align*}
\]