GELFAND TRANSFORMS AND CROFTON FORMULAS

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ABSTRACT. The term *integral geometry* has come to describe two different fields of research: one, geometrical, based on the works of Blaschke, Chern, and Santaló, and another, analytical, based on the works of Radon, John, Helgason, and Gelfand. In this paper we bridge the gap by showing that classical integral-geometric formulas such as those of Crofton, Cauchy, and Chern can be easily and systematically obtained through the study of Radon-type transforms on double fibrations. The methods also allow us to extend these formulas to non-homogeneous settings where group-theoretic techniques are no longer useful. To illustrate this point, we construct all Finsler metrics on projective space such that hyperplanes are area-minimizing and simplify the theory of Crofton densities developed by Busemann, Pogorelov, Gelfand, and Smirnov.

> *Pensar es olvidar diferencias ... .*
> Jorge Luis Borges.

CONTENTS

1. INTRODUCTION

To some mathematicians integral geometry is about beautiful geometric identities and inequalities, while to others it is about integral transforms and the study of their ranges, kernels, inversion formulas, and applications to the solution of differential equations. In this paper, we bridge the gap between both integral geometries by showing that those beautiful geometric identities can be easily and systematically obtained through the use of Radon-type transforms on double fibrations, here called *Gelfand transforms* in honor of the Gelfand school of integral geometry who introduced them in [9]. However, this paper is not so much about simplifying and

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understanding the integral geometry of Blaschke, Chern, and Santaló in terms of Gelfand transforms as it is about introducing the right concepts and tools to extend their results to cases where group-theoretic techniques no longer apply. In fact, the techniques of this paper were developed by the authors when they extended the classical Crofton formulas to all Finsler metrics on $\mathbb{R}^n$ whose geodesics are straight lines. In these spaces the length of a line segment is given by the measure of the set of all hyperplanes intersecting it. This is a classical integral-geometric result in Euclidean, elliptic, and hyperbolic geometry. The twist is that the measure on the space of hyperplanes is now arbitrary and by no means invariant under the action of some group. The main result in [2] is that for such a metric on $\mathbb{R}^n$ the area of a region lying on a k-dimensional flat equals the measure of the set of $(n-k)$-flats intersecting it. The impossibility of using classical techniques based on group invariance led us to develop new tools which turned out to be so simple that we thought of challenging A. Weil's dictum that integral geometry really belongs "within the framework of E. Cartan's theory of homogenous spaces" ([19]). Indeed, as one of our simplest applications, we show how to clarify and prove the results in the pioneering paper of Chern in such a way that frees them from all of Weil's criticisms.

The structure of the paper is as follows: section 2 contains some preliminaries such as densities and double fibrations. In section 3, we review Gelfand's construction of an integral transform associated to a double fibration. The reader may have already remarked that a large number of classical integral-geometric formulas have the following form: in a manifold B we have a family of submanifolds B_γ parametrized by the points in a second manifold Γ. If Φ is a measure on Γ, then for any submanifold $N \subset B$ whose dimension is complementary to that of the B_γ we associate the integral

$$S(N) := \int_{\gamma \in \Gamma} \#(N \cap B_\gamma)\Phi.$$

We would like to write $S(N)$ as the integral over N of some "area" integrand ϕ on B that is, of course, independent of N. The idea that the transformation $\Phi \mapsto \phi$ should be a Radon-type transform was explained to the first author by I.M. Gelfand. What surprised us was the discovery that the formula should be so simple and the proof so trivial. This formula is obtained in section 4 and forms the heart of the paper. Also in section 4, we use the formula to clarify Chern's generalized Crofton formula on homogeneous spaces and derive a number of classical applications. In section 5, we prove a useful functorial property of Gelfand transforms and use it to give a simple proof of what Chern calls generalized Cauchy formulas. Section 6 contains a short report on the theory of Crofton densities due to Busemann, Pogorelov, Gelfand, and Smirnov, while section 7 contains an application of this theory to the characterization of Finsler metrics on $\mathbb{RP}^n$ for which hyperplanes are area-minimizing.

Acknowledgments . The authors gladly acknowledge their debt to the works of Chern, Busemann, Gelfand, and Smirnov ([8, 7, 10]) and thank R. Howard and R. Schneider for their comments on an earlier version of this paper. The first author also thanks I.M. Gelfand for many enlightening conversations on integral geometry.

2. PRELIMINARIES

Densities. Roughly speaking, densities are the most general objects that can be integrated over submanifolds independently of parametrization and orientation. Since we could not find any textbook where densities were presented, we shall begin from scratch by defining densities on vector spaces and manifolds and studying some of the operations that can be performed on them.

Definition 2.1. A *k-density* ϕ on a vector space V is a smooth, real-valued function of k linearly independent vectors $v = (v_1, \ldots, v_k)$ in V such that if $v' = (v'_1, \ldots v'_k)$ is another set of k linearly independent vectors that spans the same k-dimensional subspace of V, i.e., $v' = Av$ with $A \in GL(k, \mathbb{R})$, then

$$\phi(v') = |\det A| \phi(v).$$

In other words, a k-density is an even homogeneous function of degree one defined on the Grassmann cone of decomposable k-vectors in $\Lambda^k(V)$. For example, a norm on a vector space is a 1-density. As a second example, note that a positive-definite inner product on a vector space V induces a positive-definite inner product, and hence a norm, on the exterior powers $\Lambda^k(V), 1 \leq k \leq n$. The restriction of this norm to the Grassmann cone of decomposable k-vectors in $\Lambda^k(V)$ is a k-density on V. In [13], Holmes and Thompson give an interesting generalization of this example:

Definition 2.2. Let $(V, ||\cdot||)$ be a normed space with unit ball B and let $v_1, \ldots, v_k \in V$ be k linearly independent vectors spanning a subspace W. Define a Euclidean structure on the dual space W^* by prescribing that the basis dual to $v_1, \ldots v_k$ be orthonormal. The value of the *Holmes-Thompson k-volume density* φ_k at $(v_1, \ldots, v_k)$ is defined as the Euclidean volume of $(B \cap W)^* \subset W^*$ divided by the volume of the Euclidean unit ball of dimension k.

Remark . Convex geometers will recognize the $(n-1)$-volume density as the support function of the projection body of the polar of B.

Going from densities on vector spaces to smooth densities on manifolds is as easy as going from forms to differential forms:

Definition 2.3. A smooth k-density on a manifold M is a smooth map that assings to each point $x \in M$ a k-density on the vector space $T_x M$.

Recalling that the formula for change of variables in integrals uses the absolute value of the determinant of the Jacobian, the reader will have no trouble in verifying that the integral of a smooth k-density over a k-dimensional manifold does not depend on the parametrization, nor the orientation, of the manifold. In the rest of the paper, we will refer to smooth n-densities on n-dimensional manifolds as smooth measures. These measures may, of course, be signed.

Operations on densities. The two basic operations performed on densities or forms in integral geometry are the *pull-back* by a smooth map and the *push-forward* or *fibre integration* by a submersion. In keeping with the elementary character of this work, we review the definitions:

If $f : M \to N$ is a smooth map and φ is a smooth k-density on N, then the *pull-back* $f^*\varphi$ defined by

$$f^*\varphi(m; v_1, \ldots, v_k) := \varphi(f(m); D_m f(v_1), \ldots, D_m f(v_k))$$

is a smooth k-density on M.

Remark . The pull-back of densities and the Holmes-Thompson construction allow us to define a notion of area for submanifolds of finite-dimensional normed spaces which is particularly well suited to the study of integral geometry (see [2, 3, 17, 13, 18].)

Definition 2.4. Let $f : N \to V$ be an embedding of a k-dimensional manifold into a normed space V. If φ_k denotes the Holmes-Thompson k-density on V, then we define *Holmes-Thompson area* of $f(N)$ as the integral of $f^*\varphi_k$ over N.

Let $\rho : M \to N$ be a fibration and let p be the dimension of the fibres. If φ is a k-density on M with $k \geq p$, then the *push-forward* of φ, denoted by $\rho_*\varphi$, is a $(k-p)$-density on N. In order to evaluate $\rho_*\varphi$ at the tangent vectors $v_1, \dots, v_{k-p} \in T_yN$, define at each point $x \in \rho^{-1}(y)$ a top-order density on $T_x\rho^{-1}(y)$ as follows:

- Take tangent vectors $\zeta_1, \dots, \zeta_{k-p} \in T_xM$ such that $D_x\rho(\zeta_i) = v_i, 1 \leq i \leq k-p$.
- Contract the density $\varphi(x; \cdot)$ with the $(k-p)$-vector $\zeta_1 \wedge \cdots \wedge \zeta_{k-p}$ and consider the result as a density on $T_x\rho^{-1}(y)$. This density does not depend on the choice of the ζ_i's as long as they project to the v_i's.

Performing this construction at every point x on the fibre, we obtain a p-density on $\rho^{-1}(y)$ that we shall denote by $\varphi_{v_1 \wedge \cdots \wedge v_{k-p}}$. The push-forward of φ evaluated at $(v_1, \dots, v_k)$ is, by definition, equal to the integral

$$\rho_*\varphi(y; v_1, \dots, v_{k-p}) := \int_{\rho^{-1}(y)} \varphi_{v_1 \wedge \cdots \wedge v_{k-p}}.$$

It follows from this construction that if $S \subset N$ is a submanifold of dimension $k-p$, then

$$\int_S \rho_*\varphi = \int_{\rho^{-1}(S)} \varphi.$$

Double fibrations. As we mentioned in the introduction, a key element in integral geometry is a smooth manifold B together with a family of submanifolds B_γ parametrized by the points of a second manifold Γ. It is important to note that the points of B parametrize a family of submanifolds of Γ. Indeed, if $b \in B$, we set

$$\Gamma_b := \{\gamma \in \Gamma : b \in B_\gamma\} .$$

This duality between B and Γ motivates Gelfand's definition of double fibrations:

Definition 2.5. A double fibration is a diagram of manifolds

$$\begin{tikzcd} & A \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ B & & \Gamma \end{tikzcd}$$

with the following properties:

- $\pi_1 : A \to B$, $\pi_2 : A \to \Gamma$ are fibre bundles.
- The map $\pi_1 \times \pi_2 : A \to B \times \Gamma$ is an immersion.
- For each $b \in B$ and for each $\gamma \in \Gamma$, the sets $B_\gamma := \pi_1(\pi_2^{-1}(\gamma)) \subset B$ and $\Gamma_b := \pi_2(\pi_1^{-1}(b)) \subset \Gamma$ are smooth submanifolds.

Examples.

1. Let V be an affine space, let $H_k(V), k < \dim(V)$, be the space of k-flats in V, and let $A \subset V \times H_k(V)$ be the set $\{(x, \lambda) \in V \times H_k(V) : x \in \lambda\}$. If π_1 and π_2 denote the obvious projections, the diagram

$$\begin{tikzcd} & A \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ V & & H_k(V) \end{tikzcd}$$

is a double fibration.

2. A version of the previous example, which is very useful in performing explicit computations, may be described by means of multilinear algebra:

Let $\mathcal{S}^n_{n-k}$ denote the intersection of the unit sphere in $\Lambda^{n-k}(\mathbb{R}^{n*})$ and the cone of decomposable $(n-k)$-covectors. Define π_2 as the map that assigns to each pair (x, ξ) in $\mathbb{R}^n \times \mathcal{S}^n_{n-k}$ the affine subspace

$$\pi_2(x, \xi) := \{y \in \mathbb{R}^n : \xi \rfloor (y - x) = 0\} \in H_k(\mathbb{R}^n).$$

If π_1 denotes the projection onto the first factor,

$$\begin{tikzcd} & \mathbb{R}^n \times \mathcal{S}^n_{n-k} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathbb{R}^n & & H_k(\mathbb{R}^n) \end{tikzcd}$$

is a double fibration.

3. *Chern's double fibration of homogeneous spaces.* Let G be a Lie group and let H and K be subgroups of G. If G/H and G/K are the quotient spaces, define the incidence relation $A := \{(xH, yK) \in G/H \times G/K : xH \cap yK \neq \emptyset\}$. It is not difficult to see that the diagram

$$\begin{tikzcd} & A \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ G/H & & G/K \end{tikzcd}$$

is a double fibration. Moreover, the incidence relation A can be naturally identified with the homogeneous space $G/(H \cap K)$. This double fibration was considered by Chern in his pioneering article [8].

3. GELFAND TRANSFORMS ASSOCIATED TO DOUBLE FIBRATIONS

Definition 3.1. Let $B \xleftarrow{\pi_1} A \xrightarrow{\pi_2} \Gamma$ be a double fibration. Let k be the dimension of the fibres of $\pi_1 : A \longrightarrow B$ and let Φ be an m-density on Γ with $m \geq k$. The Gelfand transform of Φ is the $(m-k)$-density $\pi_{1*}\pi_2^*\Phi$.

Classical integral-geometric transforms such as those of Funk, Radon, and John are special cases of the Gelfand transform:

The Funk transform. The Funk transform of a smooth function f on the two-sphere associates to each great circle γ the integral of $f ds$ along γ. Here ds denotes the arclength element of the standard metric on S^2. To write the Funk transform as a Gelfand transform, let S^{2*} denote the space of oriented great circles on S^2 and consider the double fibration

$$\begin{tikzcd} & A \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ S^{2*} & & S^2, \end{tikzcd}$$

where A is the canonical incidence relation. If γ is a great circle, then

$$\pi_{1*}\pi_2^*(fds)(\gamma) = \int_\gamma fds \ .$$

The Radon transform. The Radon transform of a smooth, rapidly decreasing function f on $\mathbb{R}^3$ associates to each affine plane ζ the integral of fdA over ζ. Here dA denotes the area element for the Euclidean metric on $\mathbb{R}^3$. To write the Radon transform as a Gelfand transform, let $H_2(\mathbb{R}^3)$ be the space of affine planes on $\mathbb{R}^3$ and consider the double fibration

$$\begin{tikzcd} & A \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ H_2(\mathbb{R}^3) & & \mathbb{R}^3, \end{tikzcd}$$

where A is the canonical incidence relation. If ζ is an affine plane, then

$$\pi_{1*}\pi_2^*(fdA)(\zeta) = \int_\zeta fdA \ .$$

In our next example, the computation of the Gelfand transform of a smooth measure Φ on $H_{n-k}(\mathbb{R}^n)$ associated to the double fibration $\mathbb{R}^n \xleftarrow{\pi_1} \mathbb{R}^n \times \mathcal{S}^n_k \xrightarrow{\pi_2} H_{n-k}(\mathbb{R}^n)$ is made trivial by the following remark: if Ω is the standard volume form on $\mathcal{S}^n_k$ (i.e., the volume form induced from its embedding in the Euclidean space $\Lambda^k(\mathbb{R}^{n*})$), then there exists a unique function ν on $H_{n-k}(\mathbb{R}^n)$ satisfying $(\pi_2^*\Phi)_{(x,\xi)} = \nu(\pi_2(x,\xi))|\xi \wedge \Omega|$ for all $(x,\xi) \in \mathbb{R}^n \times \mathcal{S}^n_k$.

Proposition 3.1. *Consider the double fibration* $\mathbb{R}^n \xleftarrow{\pi_1} \mathbb{R}^n \times \mathcal{S}^n_k \xrightarrow{\pi_2} H_{n-k}(\mathbb{R}^n)$ *and let* Φ *be a smooth measure on* $H_{n-k}(\mathbb{R}^n)$. *If* Ω *is the standard volume form on* $\mathcal{S}^n_k$ *and* ν *is the unique smooth function on* $H_{n-k}(\mathbb{R}^n)$ *satisfying* $(\pi_2^*\Phi)_{(x,\xi)} = \nu(\pi_2(x,\xi))|\xi \wedge \Omega|$ *for all* $(x,\xi) \in \mathbb{R}^n \times \mathcal{S}^n_k$*, then for any* k*-vector* $v \in \Lambda^k(T_x\mathbb{R}^n)$

$$\pi_{1*}\pi_2^*\Phi(x,v) = \int_{\xi \in \mathcal{S}^n_k} |\xi \cdot v| \nu(\pi_2(x,\xi))\Omega.$$

A note on the cosine transform.

Note that if the measure Φ in proposition 3.1 is invariant under translations, then

$$\pi_{1*}\pi_2^*\Phi(v) = \int_{\xi \in \mathcal{S}^n_k} |\xi \cdot v| \nu(\xi)\Omega, \tag{1}$$

where ν is a smooth even function on $\mathcal{S}^n_k$. This transform is known as the *cosine transform* of the measure $\nu|\Omega|$ on $\mathcal{S}^n_k$.

An elementary, but extremely important observation due to Goncharov and Koldobsky ([11] and [15]) is that the cosine transform for $k = 1$ is just a different way of writing the inverse Fourier transform for homogeneous functions of degree one:

Let ϕ be a smooth, even homogeneous function of degree one on an n-dimensional vector space V, let $e_1, \ldots, e_n$ be a basis of V, and let $\xi_1, \ldots, \xi_n$ be the dual basis in V^*. The basis $e_1, \ldots, e_n$ allows us to identify both V and V^* with $\mathbb{R}^n$ and hence to compute the (distributional) Fourier transform

$$\widehat{\phi}(\xi) := \int_{\mathbb{R}^n} e^{i\xi\cdot v}\phi(v)dv.$$

While this transform depends on the Lebesgue measure associated to the chosen basis, the form $\widehat{\phi}\, d\xi_1 \wedge \cdots \wedge d\xi_n$ does not. Up to a constant factor, we define the *Fourier transform* of ϕ as the contraction of this n-form with the Euler vector field, $X_E(\xi) = \xi$, in V^*:

$$\check{\phi} := \frac{-1}{4(2\pi)^{n-1}}\widehat{\phi}\, d\xi_1 \wedge \cdots \wedge d\xi_n \rfloor X_E.$$

It is known (see [14], pages 167-168) that $\widehat{\phi}$ is smooth on $V^* \setminus \{0\}$ and homogeneous of degree $-n-1$. It follows that $\check{\phi}$ is a smooth differential form on $V^* \setminus \{0\}$ which is homogeneous of degree -1.

If $V = \mathbb{R}^n$, then the pull-back of $\check{\phi}$ to S^{n-1} can be written as $\nu\,\Omega$, where ν is a smooth function and Ω is the standard volume form.The cosine transform allows us to recover ϕ from $\check{\phi}$:

$$\phi(v) := \int_{\xi\in S^{n-1}} |\xi\cdot v|\nu(\xi)\Omega\ . \tag{2}$$

Since the same arguments can be applied when $k = n-1$, we have the following:

Proposition 3.2. *For $k = 1, n-1$, the cosine transform is a bijection between smooth even measures on $\mathcal{S}^n_k$ and k-densities on $\mathbb{R}^n$.*

By identifying $T_x\mathbb{R}^n, x \in \mathbb{R}^n$, with $\mathbb{R}^n$, it follows that smooth k-densities, $k = 1, n-1$, on $\mathbb{R}^n$ have a natural integral representation.

Corollary 3.1. *If φ is a smooth k-density on $\mathbb{R}^n$ with $k = 1, n-1$, then there exists a unique smooth function $\rho(x,\xi)$ on $\mathbb{R}^n \times \mathcal{S}^n_k$ that is even in ξ and satisfies*

$$\varphi(x,v) = \int_{\mathcal{S}^n_k} |\xi\cdot v|\rho(x,\xi)\Omega,$$

where Ω is the standard volume form on the unit sphere $\mathcal{S}^n_k \subset \Lambda^k(\mathbb{R}^{n*})$.

We remark that for $1 < k < n-1$, the transformation that takes a smooth even measure μ on $\mathcal{S}^n_k$ to the k-density

$$\varphi(v) := \int_{\xi\in\mathcal{S}^n_k} |\xi\cdot v|d\mu(\xi)$$

is neither injective nor surjective (see [12]).

4. Gelfand transforms and Crofton formulas

Let B be an n-dimensional manifold provided with a family of $(n-k)$-dimensional submanifolds parametrized by the points of a second manifold Γ. If Φ is measure on Γ, then we may define the following functional on the space of k-dimensional submanifolds of B:

$$S_\Phi(N) := \int_{\gamma\in\Gamma} \#(B_\gamma \cap N)\Phi \ .$$

A Crofton formula is an expression of the form

$$\int_N \varphi = \int_{\gamma\in\Gamma} \#(B_\gamma \cap N)\Phi \ ,$$

where the integrand φ does not depend on the choice of N. It is remarkable that Crofton formulas do exist at this level of generality and that they take the following simple form:

Theorem 4.1. *Let* $B \xleftarrow{\pi_1} A \xrightarrow{\pi_2} \Gamma$ *be a double fibration with* $dim(B) = n$ *and* $dim(B_\gamma) = n-k$. *If* Φ *is a smooth measure on* Γ *and* $N \subset B$ *is an immersed submanifold of dimension* k*, then*

$$S_\Phi(N) = \int_\Gamma \#(N \cap B_\gamma)\Phi = \int_N \pi_{1*}\pi_2^*\Phi.$$

The proof depends on the simplest case of the coarea formula:

Lemma 4.1. *Let* X *and* Y *be two manifolds of the same dimension, let* $f : X \to Y$ *be a smooth map, and let* Φ *be a top-order density on* Y*. If for every regular value of* f *the number of preimages is finite, then*

$$\int_X f^*\Phi = \int_{y\in Y} \#(f^{-1}(y))\Phi.$$

Proof of theorem 4.1. Using the definition of push-forwards, we have that

$$\int_N \pi_{1*}\pi_2^*\Phi = \int_{\pi_1^{-1}(N)} \pi_2^*\Phi.$$

If we now apply the coarea with $X := \pi_1^{-1}(N)$, $Y := \Gamma$, and $f := \pi_2$ while noticing that $\#(\pi_2^{-1}(\gamma) \cap \pi_1^{-1}(N)) = \#(N \cap B_\gamma)$, we obtain

$$\int_N \pi_{1*}\pi_2^*\Phi = \int_\Gamma \#(N \cap B_\gamma)\Phi.$$

□

As a first, and nearly trivial, application let us rederive and clarify Chern's generalized Crofton formula for homogeneous spaces. An advantage of our approach is that the integrand in the right-hand side, Chern's mysterious *k-area*, is defined explicitly, while in Chern's approach "the treatment is exceedingly sketchy at this point" (see A. Weil's review of Chern's article in [19]).

Theorem 4.2 (Chern). *Let* G *be a Lie group, let* H *and* K *be closed subgroups of* G*, let* n *be the dimension of* $G/(H \cap K)$*, and let* $n-k$ *be the dimension of the fibres of* $\pi_1 : G/(H \cap K) \to G/H$*. If* G/K *admits an invariant measure* Φ

and H is compact, then there exists an invariant k-density φ such that for every k-dimensional submanifold $N \subset G/H$

$$\int_{yK \in G/K} \#(N \cap B_{yK})\Phi = \int_N \varphi \ ,$$

where $B_{yK} := \{xH \in G/H : xH \cap yK \neq \emptyset\}$.

Proof. Simply apply the Crofton formula for double fibrations to the double fibration

$$\begin{tikzcd} & A \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ G/H & & G/K \end{tikzcd}$$

and the invariant measure Φ. The k-density φ, Chern's elusive *k-area*, turns out to be nothing more than $\pi_{1*}\pi_2^*\Phi$ and it is invariant because Φ is invariant and the whole diagram is equivariant. ☐

In most applications of Chern's theorem, computations are avoided by noticing the uniqueness of invariant k-densities under various hypotheses. In other words, we don't have to compute the k-area because, up to multiples, there is only one and it is already known to be the area element for an invariant Riemannian metric on the homogeneous space. Below, we make this idea precise and give a sample of interesting applications.

If B is a homogeneous space under the left action of a Lie group G and k is an integer with $1 \leq k < \dim(B)$, the action of G on B induces a left action of G on the *Grassmann bundle* $Gr_k(TB)$ of k-planes on TB.

Lemma 4.2. *Let B and G be as above and let $E \subset Gr_k(TB)$ be a subbundle which is invariant under the action of G and on which this action is transitive. If ψ_k and φ_k are two G-invariant k-densities on B, then there exists a constant c such that whenever $v_1, \ldots, v_k \in T_xB$ span a k-plane in E, then $\varphi_k(x; v_1, \ldots, v_k) = c\,\psi_k(x; v_1, \ldots, v_k)$.*

For elliptic, Euclidean, or hyperbolic n-space, the subbundle E is all of $Gr_k(TB)$. This means that for every k, $1 \leq k \leq n$, there is, up to a multiple, a unique k-density that is invariant under the actions of the isometry group. Since the k-area density, vol_k, for the Riemannian metric is invariant, any other invariant k-density is a multiple of it. This remark together with Chern's formula immediately implies the following result:

Theorem 4.3. *Let B denote either the elliptic, Euclidean, or hyperbolic n-space and let G be its group of isometries. If $\Gamma_k, 1 \leq k < n$, denotes the space of complete, totally geodesic, k-dimensional submanifolds of B and Φ_k is a G-invariant measure on Γ_k, then there exists a constant $c_{n,k}$ such that for every smooth $(n-k)$-dimensional submanifold $N \subset B$*

$$vol_{n,k}(N) = c_{n,k} \int_{\gamma \in \Gamma_k} \#(N \cap \gamma)\Phi_k \ .$$

As an application that makes better use of the generality of lemma 4.2, let us consider the complex projective n-space $\mathbb{CP}^n$ with its Fubini-Study metric. The group of isometries in this case is $U(n+1)$, which does not act transitively on the

Grassmann bundles. However, for each even number $2k$, $1 \leq k < n$, $U(n+1)$ does act transitively on the subbundle of complex tangent subspaces of complex dimension k. As a consequence, the integral of a $U(n+1)$-invariant $2k$-density over any complex submanifold of complex dimension k is equal to a fixed multiple of the area of the submanifold.

Note that the group $U(n+1)$ also acts transitively on the Grassmann bundle of tangent Lagrangian subspaces of $\mathbb{CP}^n$. By lemma 4.2 the integral of a $U(n+1)$-invariant n-density over any Lagrangian submanifold is equal to a fixed multiple of the area of the submanifold.

Recalling that the complex Grassmannian $Gr_{k+1}(\mathbb{C}^{n+1})$ parametrizes the space of complex projective subspaces of dimension k on $\mathbb{CP}^n$ and that the Lagrangian Grassmannian $\Lambda_{n+1}(\mathbb{C}^{n+1})$ parametrizes the space of totally geodesic (Lagrangian) $\mathbb{RP}^n$'s on $\mathbb{CP}^n$, we have the following immediate corollaries of Chern's formula:

Theorem 4.4. *If Φ_k is a $U(n+1)$-invariant measure on the complex Grassmannian $Gr_{k+1}(\mathbb{C}^{n+1})$, then there exists a constant $c_{n,k}$ such that for every $(n-k)$-dimensional complex submanifold $N \subset \mathbb{CP}^n$*

$$vol_{2(n-k)}(N) = c_{n,k} \int_{\zeta \in Gr_{k+1}(\mathbb{C}^{n+1})} \#(N \cap \zeta)\Phi_k \ .$$

Theorem 4.5. *If Φ is a $U(n+1)$-invariant measure on the Grassmannian of Lagrangian planes in $\mathbb{C}^{n+1}$, $\Lambda_{n+1}(\mathbb{C}^{n+1})$, then there exists a constant c_n such that for every Lagrangian submanifold $L \subset \mathbb{CP}^n$*

$$vol_n(L) = c_n \int_{\lambda \in \Lambda_{n+1}(\mathbb{C}^{n+1})} \#(L \cap \lambda)\Phi \ .$$

5. A functorial property of the Gelfand transform

Generically, a k-flat and a p-flat in an n-dimensional vector space intersect along an $(k+p-n)$-flat. Morever, if the point x is incident to both the k-flat and the p-flat, then it is also incident to their intersection. This trivial remark suggests the following definition for a morphism of double fibrations.

Definition 5.1. A morphism between two double fibrations $B' \xleftarrow{\pi_1{}'} A' \xrightarrow{\pi_2{}'} \Gamma'$ and $B \xleftarrow{\pi_1} A \xrightarrow{\pi_2} \Gamma$ is a commutative diagram of fibrations

$$\begin{tikzcd}
B' \arrow[d, "\rho_B"'] & A' \arrow[l, "\pi_1'"'] \arrow[r, "\pi_2'"] \arrow[d, "\rho_A"'] & \Gamma' \arrow[d, "\rho_\Gamma"'] \\
B & A \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & \Gamma .
\end{tikzcd}$$

In the example we have in mind, Γ' is the spaces of pairs consisting of a k-flat and a p-flat in $\mathbb{R}^n$ that intersect transversally. The incidence relation A' is made up of triples consisting of a point x and a pair $(\lambda_1, \lambda_2) \in \Gamma'$ such that x belongs to both λ_1 and λ_2. In the second double fibration, Γ is the space of $(k+p-n)$-flats, while A is the standard incidence relation. The projection ρ_Γ sends a pair (λ_1, λ_2) to their intersection, ρ_A sends a triple $(x, \lambda_1, \lambda_2)$ to $(x, \lambda_1 \cap \lambda_2)$, and ρ_B is the identity map.

The main result of this section is the following functorial property of Gelfand transforms.

Theorem 5.1. *Let*

$$\begin{tikzcd} B' \arrow[d, "\rho_B"'] & A' \arrow[l, "\pi_1'"'] \arrow[r, "\pi_2'"] \arrow[d, "\rho_A"'] & \Gamma' \arrow[d, "\rho_\Gamma"'] \\ B & A \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & \Gamma \end{tikzcd}$$

be a morphism of double fibrations such that for every point $a \in A$ the map π_2' restricted to $\rho_A^{-1}(a)$ is a diffeomorphism onto $\rho_\Gamma^{-1}(\pi_2(a))$. If Φ is a density Γ', then

$$\rho_{B*}\pi'_{1*}(\pi_2')^*\Phi = \pi_{1*}\pi_2^*\rho_{\Gamma*}\Phi.$$

Proof. Since $\rho_B \circ \pi_1' = \pi_1 \circ \rho_A$ and, therefore, $\rho_{B*}\pi'_{1*} = \pi_{1*}\rho_{A*}$, it is enough to consider the diagram

$$\begin{tikzcd} A' \arrow[r, "\pi_2'"] \arrow[d, "\rho_A"'] & \Gamma' \arrow[d, "\rho_\Gamma"'] \\ A \arrow[r, "\pi_2"] & \Gamma \end{tikzcd}$$

and show that $\pi_2^*\rho_{\Gamma*}\Phi = \rho_{A*}(\pi_2')^*\Phi$.

In order to do this, denote by k the order of $\pi_2^*\rho_{\Gamma*}\Phi$ and let $v_1, \dots, v_k$ be k vectors on T_aA. Recall from section 2 the construction of the density $((\pi_2')^*\Phi)_{v_1\wedge\cdots\wedge v_k}$ defined on the fibre $\rho_A^{-1}(a)$: at each point $a' \in \rho_A^{-1}(a)$ choose k vectors $\zeta_1, \dots, \zeta_k$ which project onto $v_1, \dots, v_k$ and set

$$((\pi_2')^*\Phi)_{v_1\wedge\cdots\wedge v_k}(\cdot) = ((\pi_2')^*\Phi)(\zeta_1, \dots, \zeta_k, \cdot).$$

On the other hand, if $w_1, \dots, w_k$ denote the images of $v_1, \dots, v_k$ under the differential of π_2, then at the point $\pi_2'(a') \in \rho_\Gamma^{-1}(\pi_2(a))$ the density $\Phi_{w_1\wedge\cdots\wedge w_k}$ can be defined by

$$\Phi_{w_1\wedge\cdots\wedge w_k}(\cdot) = \Phi(D\pi_2'(\zeta_1), \dots, D\pi_2'(\zeta_k), \cdot).$$

This is simply because $\pi_2 \circ \rho_A = \rho_\Gamma \circ \pi_2'$. We conclude that

$$((\pi_2')^*\Phi)_{v_1\wedge\cdots\wedge v_k} = (\pi_2')^*\Phi_{w_1\wedge\cdots\wedge w_k}.$$

Since, by hypothesis, π_2' is a diffeomorphism from $\rho_A^{-1}(a)$ to $\rho_\Gamma^{-1}(\pi_2(a))$ we have that

$$\begin{aligned}\rho_{A*}(\pi_2')^*\Phi(v_1, \dots, v_k) &= \int_{\rho_A^{-1}(a)} ((\pi_2')^*\Phi)_{v_1\wedge\cdots\wedge v_k} = \int_{\rho_A^{-1}(a)} (\pi_2')^*\Phi_{w_1\wedge\cdots\wedge w_k} \\ &= \int_{\rho_\Gamma^{-1}(\pi_2(a))} \Phi_{w_1\wedge\cdots\wedge w_k} = \pi_2^*\rho_{\Gamma*}\Phi(v_1, \dots, v_k).\end{aligned}$$

□

An easy application of the preceding theorem is the following generalized Cauchy formula for the classical geometries.

Theorem 5.2. *Let B denote either the elliptic, Euclidean, or hyperbolic n-space and let G be its group of isometries. Moreover, let $\Gamma_p, 1 \le p < n$, denote the space of complete, totally geodesic, p-dimensional submanifolds of B and let Φ_p be a G-invariant measure on Γ_p. If k is an integer satisfying $1 \le k < n$ and $k + p > n$, then there exists a constant $c = c_{n,k,p}$ such that for every k-dimensional submanifold $N \subset B$*

$$vol_k(N) = c \int_{\gamma\in\Gamma_p} vol_{k+p-n}(N \cap \gamma)\Phi_p \ .$$

The reason Chern called this type of formula a *generalized Cauchy formula* is because of a classical theorem of Cauchy stating that the integral of the lengths of the intersections of all planes with a fixed surface in Euclidean 3-space equals $\pi^2/2$ times the area of the surface. The reader will have no trouble deducing this result from the theorem above.

Proof. In order to faciliate the notation, let us assume that the invariant measures on the spaces of totally geodesics submanifolds of B have been normalized so that the Crofton formula

$$vol_m(M) = \int_{\gamma \in \Gamma_{n-m}} \#(M \cap B_\gamma)\Phi_{n-m}$$

holds.

The idea of the proof is to apply theorem 5.1 to a well-chosen morphism of double fibrations. Using notation similar to that introduced at the begining of this section, this morphism is:

$$\begin{tikzcd}
B & A' \arrow[l, "\pi'_1"'] \arrow[r, "\pi'_2"] \arrow[d, "\rho_A"'] & \Gamma' \arrow[d, "\rho_\Gamma"'] \\
B \arrow[u, phantom] & A_{n-k} \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & \Gamma_{n-k},
\arrow[from=1-1, to=2-1, "id"']
\end{tikzcd}$$

where Γ' is the space of pairs consisting of a p-dimensional and a $(2n-k-p)$-dimensional totally geodesic submanifold in B that intersect transversally. The incidence relation A' is made up of triples consisting of a point x and a pair $(\lambda_1, \lambda_2) \in \Gamma'$ such that x belongs to both λ_1 and λ_2. In the second double fibration, Γ_{n-k} is the space of $(n-k)$-dimensional totally geodesic submanifolds, while A_{n-k} is the standard incidence relation. The projection ρ_Γ sends a pair (λ_1, λ_2) to their intersection and ρ_A sends a triple $(x, \lambda_1, \lambda_2)$ to $(x, \lambda_1 \cap \lambda_2)$. It is easy to verify that this morphism of double fibrations satisfies the hypothesis in 5.1 and, therefore, $\pi_{1*}\pi_2^*\rho_{\Gamma *} = \pi'_{1*}(\pi'_2)^*$.

Note that $\rho_{\Gamma *}(\Phi_p \times \Phi_{2n-k-p})$ is an invariant measure on Γ_{n-k} and that, by lemma 4.2, it is equal to a constant c^{-1} times Φ_{n-k}.

$$\begin{aligned}
vol_k(N) &= \int_N \pi_{1*}\pi_2^*\Phi_{n-k} = c\int_N \pi'_{1*}(\pi'_2)^*\Phi_p \times \Phi_{2n-k-p} = \\
c\int_{\Gamma'} &\#(N \cap \gamma \cap \lambda)\Phi_p \times \Phi_{2n-k-p} = c\int_{\Gamma_p}\int_{\Gamma_{2n-k-p}} \#((N \cap \gamma) \cap \lambda)\Phi_{2n-k-p}\Phi_p \\
&= c\int_{\gamma \in \Gamma_p} vol_{k+p-n}(N \cap \gamma)\Phi_p.
\end{aligned}$$

□

Clearly, the proof above can be easily extended to many different situations. A noteworthy example is the following result:

Theorem 5.3. *Let Φ_p be a $U(n+1)$-invariant measure on the Grassmannian $Gr_{p+1}(\mathbb{C}^{n+1})$ of complex $(p+1)$-planes on $\mathbb{C}^{n+1}$. If k is an integer satisfying $1 \le k < n$ and $k + p > n$, then there exists a constant $c = c_{n,k,p}$ such that for every k-dimensional complex submanifold $N \subset \mathbb{CP}^n$*

$$vol_{2k}(N) = c\int_{\gamma \in Gr_{p+1}(\mathbb{C}^{n+1})} vol_{2(k+p-n)}(N \cap \gamma)\Phi_p \ .$$

6. CROFTON DENSITIES AND PROJECTIVE DENSITIES

The simplest of all inverse problems in the calculus of variations is to determine all variational problems on $\mathbb{R}^n$ for which k-flats are extremals. The problem has been addressed by Hilbert, Hamel, Funk, Busemann, Pogorelov, and Gelfand, among others. In this section we give a two-page proof of the main result in the area.

Definition 6.1. A smooth k-density φ on an open convex subset $\mathcal{O} \subset \mathbb{RP}^n$ is called *projective* if the intersection of $\mathcal{O}$ with any k-dimensional projective subspace extremizes the variational problem $N \mapsto \int_N \varphi$.

Definition 6.2 (Gelfand and Smirnov, [10]). A smooth k-density φ defined on an open convex subset $\mathcal{O} \subset \mathbb{RP}^n$ is a *Crofton density* if there exists a smooth measure Φ defined on the set of $(n-k)$-dimensional projective subspaces passing through $\mathcal{O}$, $H_{n-k}(\mathcal{O})$, such that

$$\int_N \varphi = \int_{\lambda \in H_{n-k}(\mathcal{O})} \#(\lambda \cap N)\Phi$$

for any smooth k-dimensional submanifold $N \subset \mathcal{O}$.

Theorem 6.1. *Crofton densities are projective. Moreover, for $k = 1, n-1$ projective k-densities are Crofton densities.*

The case $k = 1$ is due to Pogorelov (see [16]). The higher dimensional case should probably be attributed to Gelfand and Smirnov, although they incorrectly state in theorem 4 of [10] that a density is projective if and only if it is Crofton and their proof does not seem to be complete.

The following result puts the proof of theorem 6.1 in perspective and illustrates the idea of Busemann (see [7]) that inspired Pogorelov.

Theorem 6.2. *If the defining measure of a Crofton k-density φ is non-negative, then k-dimensional subspaces are area-minimizing. Moreover, if the density is defined on all of $\mathbb{RP}^n$, then k-dimensional subspaces are absolutely minimizing in their homology class.*

Proof. If D is a domain in a k-dimensional projective subspace and D' is another k-dimensional submanifold with the same boundary, then any k-dimensional projective subspace intersecting D necessarily intersects D'. We conclude that

$$\int_D \varphi = \int_{\lambda \in H_k(\mathcal{O})} \#(\lambda \cap D)\Phi \leq \int_{\lambda \in H_k(\mathcal{O})} \#(\lambda \cap D')\Phi = \int_{D'} \varphi.$$

To prove the second part of the theorem, just note that if N is homologous to a k-dimensional projective subspace, then it intersects every $(n-k)$-dimensional projective subspace at least once. ☐

If Φ takes negative values, the preceding proof breaks down and we must abandon geometry for analysis. A key point is the following integral representation for Crofton densities. The notation is that of proposition 3.1.

Lemma 6.1. *A smooth k-density φ on $\mathbb{R}^n$ is a Crofton density if and only if it has an integral representation of the form*

$$\varphi(x, v) := \frac{1}{2} \int_{\xi \in \mathcal{S}^n_k} |\xi \cdot v| \nu(\pi_2(x, \xi)) \, \Omega \ ,$$

where $\mathcal{S}_k^n$ is the intersection of the unit sphere in $\Lambda^k(\mathbb{R}^{n*})$, Ω is the standard volume form on $\mathcal{S}_k^n$, and ν is a smooth function on $H_{n-k}(\mathbb{R}^n)$.

Proof. The Crofton formula for double fibrations tells us that

$$\int_{\lambda\in H_{n-k}(\mathbb{R}^n)} \#(\lambda\cap N)\Phi = \frac{1}{2}\int_N \pi_{1*}\pi_2^*\Phi,$$

where the one-half is explained by the fact that $\mathbb{R}^n\times\mathcal{S}_k^n$ is a double cover of the incidence relation $(n-k)$-flat-point with the point lying on the flat. In other words, the Gelfand transform counts every $(n-k)$-flat twice.

To conclude the proof, recall that proposition 3.1 states that

$$\pi_{1*}\pi_2^*\Phi(x,v) = \int_{\xi\in\mathcal{S}_k^n} |\xi\cdot v|\nu(\pi_2(x,\xi))\Omega,$$

where ν satisfies $(\pi_2^*\Phi)_{(x,\xi)} = \nu(\pi_2(x,\xi))|\xi\wedge\Omega|$. ☐

Example ([2]). Applying the previous lemma to the function $\nu(\pi_2(x,\xi)) := 1+(\xi\cdot x)^2$ defined on $\mathbb{R}^2\times S^1$, we obtain the projective 1-density

$$\varphi(x_1,x_2,v_1,v_2) = \frac{1}{3\sqrt{v_1^2+v_2^2}}\,[(3+x_1^2+x_2^2)(v_1^2+v_2^2)+(x_1v_1+x_2v_2)^2].$$

The next step in the proof is the following result which shows that projective k-densities satisfy a system of partial differential equations.

Lemma 6.2. *A smooth k-density φ, with $1\le k\le n$, in $\mathbb{R}^n$ is projective if and only if it satisfies the following system of equations:*

$$-\frac{\partial\varphi}{\partial x^l}+\sum_{p=1}^k\sum_{i=1}^n v_p^i\frac{\partial^2\varphi}{\partial x^i\partial v_p^l}=0,\ \textit{with } l=1,2,\ldots,n. \tag{3}$$

Proof. Let's consider the multidimensional variational problem associated to the functional

$$S(N)=\int_N\varphi,$$

where N is a k-dimensional submanifold of $\mathbb{R}^n$. The Euler-Lagrange equations for this variational problem are

$$-\frac{\partial\varphi}{\partial x^l}(x(t);\frac{\partial x}{\partial t^1},\ldots,\frac{\partial x}{\partial t^k})+\sum_{p=1}^k\frac{\partial}{\partial t^p}\left(\frac{\partial\varphi}{\partial v_p^l}(x(t);\frac{\partial x}{\partial t^1},\ldots,\frac{\partial x}{\partial t^k})\right)=0, \tag{4}$$

with $l=1,2,\ldots,n$, and $x(t)=(x^1(t^1,\ldots,t^k),\ldots,x^n(t^1,\ldots,t^k))$ is a parametrization of N. If we develop the terms in (4), the Euler-Lagrange equations can be rewritten as

$$-\frac{\partial\varphi}{\partial x^l}+\sum_{p=1}^k\sum_{i=1}^n\frac{\partial x^i}{\partial t^p}\cdot\frac{\partial^2\varphi}{\partial x^i\partial v_p^l}+\sum_{p=1}^k\sum_{j=1}^k\sum_{i=1}^n\frac{\partial^2x^i}{\partial t^p\partial t^j}\cdot\frac{\partial^2\varphi}{\partial v_j^i\partial v_p^l}=0, \tag{5}$$

with $l=1,2,\ldots,n$. Notice that if φ is a projective k-density, then submanifolds having a parametrization of the form $x(t)=A\cdot t+B$, with $A\in\mathbb{R}^{n\times k}$ and $B\in\mathbb{R}^{n\times 1}$ constant matrices, satisfy equations (5). This implies that

$$-\frac{\partial\varphi}{\partial x^l}+\sum_{p=1}^k\sum_{i=1}^n v_p^i\frac{\partial^2\varphi}{\partial x^i\partial v_p^l}=0,\quad\text{with } l=1,2,\ldots,n.$$

Conversely, if φ is a k-density satisfying equations (3), then submanifolds N with parametrization $x(t) = A\cdot t+B$ clearly satisfy (5) and hence are extremals of φ. ☐

Proof of theorem 6.1. Differentiating under the integral sign, it is easy to see that a density of the form

$$\varphi(x,v) := \frac{1}{2}\int_{\xi\in\mathcal{S}^n_k} |\xi\cdot v|\nu(\pi_2(x,\xi))\,\Omega\ ,$$

satisfies equation (3) . It follows from the lemmas that Crofton densities are projective.

To prove the second part of the theorem, recall that, by corollary 3.1, if $k = 1, n-1$, any k-density on $\mathbb{R}^n$ has an integral representation of the form

$$\varphi(x,v) = \int_{\mathcal{S}^n_k} |\xi\cdot v|\rho(x,\xi)\Omega.$$

Differentiating under the integral sign we see that φ satisfies (3) if and only if there exists a function ν on $H_{n-k}(\mathbb{R}^n)$ such that $\rho = \nu\circ\pi_2$. It follows that φ is a Crofton density. ☐

While we have dealt exclusively with smooth densities on $\mathbb{R}^n$, it is easy to extend the proofs to the case of smooth densities on $\mathbb{RP}^n$ by the following trick:

Let $\zeta\subset\mathbb{RP}^n$ be a projective hyperplane and let $i_\zeta : \mathbb{R}^n \mapsto \mathbb{RP}^n\setminus\zeta$ be the usual affine parametrization. Since affine subspaces on $\mathbb{R}^n$ are taken to projective subspaces on $\mathbb{RP}^n\setminus\zeta$, we have that φ is a projective (resp. Crofton) density on $\mathbb{RP}^n$ if and only if for every hyperplane ζ its pull-back, $i^*_\zeta\varphi$, to $\mathbb{R}^n$ is also a projective (resp. Crofton) density.

7. METRICS IN WHICH HYPERPLANES MINIMIZE AREA

In this section, we establish a correspondence between Finsler metrics on $\mathbb{RP}^n$ such that projective hyperplanes absolutely minimize the Holmes-Thompson area in their homology class and smooth positive measures on the Grassmannian $Gr_2(\mathbb{R}^{n+1})$. This solves problem 19 in [1], which was motivated by Bekkar and Bryant's study ([4, 5]) of Riemannian metrics on open subsets of $\mathbb{RP}^3$ for which planes are minimal surfaces.

The reader not familiar with Finsler geometry or the Holmes-Thompson area need not abandon us now: we shall review and motivate the basic definitions.

Roughly speaking, a Finsler manifold is a smooth manifold together with a choice of norm on each tangent space. However, there is an important technical assumption on these norms:

Definition 7.1. A norm $\varphi : V \to [0,\infty)$ is said to be a *Minkowski norm* if the unit spheres in the normed space (V,φ) and its dual (V^*,φ^*) are smooth.

Definition 7.2. Let M be a smooth manifold and let $TM\setminus 0$ denote its tangent bundle with the zero section deleted. A *Finsler metric* on M is a smooth function

$$\varphi : TM\setminus 0 \longrightarrow \mathbb{R}$$

such that for each point $m\in M$ the restriction of φ to T_mM is a Minkowski norm.

If (M,φ) is a Finsler manifold and $\gamma : [a,b] \to M$ is a smooth curve, we define

$$\text{length of } \gamma := \int_a^b \varphi(\dot\gamma(t))\,dt.$$

If x and y are two points on M we define their distance as the infimum of the lengths of all smooth curves joining them. Thus, Finsler manifolds are metric spaces. While this makes it possible to define the volume of a Finsler manifold as its Hausdorff measure, we will investigate an alternative definition due to Holmes and Thompson (see [13] and [18]) and whose ties with convex, integral, and symplectic geometry make it a more interesting object of study.

Definition 7.3. Let (M,φ) be an n-dimensional Finsler manifold. The *Holmes-Thompson k-volume density* of M, denoted by φ_k, is the map that assigns to each point $x \in M$ the Holmes-Thompson k-volume density of the Minkowski space $(T_xM, \varphi_{|T_xM})$. The k-volume of a k-dimensional submanifold $N \subset M$ is defined as the integral of φ_k over N.

Recalling that the Holmes-Thompson $(n-1)$-volume density of an n-dimensional normed space is the projection body of the polar of its unit ball, we have the following integral representation (see [6] formula (1.4) and theorem 3).

Theorem 7.1. *Let φ be a Minkowski norm on $\mathbb{R}^n$ and let $\hat\varphi$ be its $(n-1)$-volume density. There exists a unique smooth, even, positive measure Ψ on $\mathcal{S}^n_{n-1}$ such that*

$$\hat\varphi(v_1 \wedge \cdots \wedge v_{n-1}) = \int_{\mathcal{S}^n_{n-1}} |\xi_1 \wedge \cdots \wedge \xi_{n-1} \cdot v_1 \wedge \cdots \wedge v_{n-1}|\Psi. \qquad (6)$$

Conversely, if Ψ is a smooth, even, positive measure on $\mathcal{S}^n_{n-1}$, then there exits a unique Minkowski norm φ such that equation (6) holds.

We are now ready to explain the correspondence between smooth positive measures on the Grassmannian $Gr_2(\mathbb{R}^{n+1})$, tought as the space of projective lines on $\mathbb{RP}^n$, and Finsler metrics on $\mathbb{RP}^n$ for which hyperplanes minimize area in their homology class.

Let Φ be a smooth positive measure on $Gr_2(\mathbb{R}^{n+1})$ and consider the double fibration $\mathbb{RP}^n \xleftarrow{\pi_1} A \xrightarrow{\pi_2} Gr_2(\mathbb{R}^{n+1})$, where A is the incidence relation point-line with the point lying on the line. By the results of the previous section, $\pi_{1*}\pi_2^*\Phi$ is an $(n-1)$-density for which hyperplanes are absolutely minimizing in their homology class.

Theorem 7.2. *If Φ is a smooth positive measure on $Gr_2(\mathbb{R}^{n+1})$, then there exists a unique Finsler metric on $\mathbb{RP}^n$ such that its $(n-1)$-volume density is $\pi_{1*}\pi_2^*\Phi$. Conversely, if φ is a Finsler metric on $\mathbb{RP}^n$ for which hyperplanes minimize area, then there exists a unique smooth positive measure Φ on $Gr_2(\mathbb{R}^{n+1})$ such that the $(n-1)$-volume density of $(\mathbb{RP}^n,\varphi)$ is $\pi_{1*}\pi_2^*\Phi$.*

Proof. Let Φ be a smooth positive measure on $Gr_2(\mathbb{R}^{n+1})$. By passing to affine coordinates on $\mathbb{RP}^n$, we can write

$$\pi_{1*}\pi_2^*\Phi(x,v) = \int_{\mathcal{S}^n_{n-1}} |\xi \cdot v|\rho(x,\xi)\Omega,$$

where Ω is the standard volume form on $\mathcal{S}^n_{n-1}$ and ρ is the unique smooth function satisfying $(\pi_2^*\Phi)_{(x,\xi)} = \rho(x,\xi)|\xi \wedge \Omega|$.

Fixing x and applying theorem 7.1, we have that there exists a unique Minkowski norm φ_x on $T_x(\mathbb{RP}^n)$ such that its $(n-1)$-volume density agrees with $\pi_{1*}\pi_2^*\Phi(x,\cdot)$. By doing this at each point of $\mathbb{RP}^n$, we obtain the Finsler metric whose $(n-1)$-volume density is $\pi_{1*}\pi_2^*\Phi$.

To prove the second part of the theorem, note that if φ is a Finsler metric on $\mathbb{RP}^n$ such that hyperplanes minimize area, then theorem 6.1 implies that its $(n-1)$-volume density is a Crofton density and hence of the form $\pi_{1*}\pi_2^*\Phi$ for some smooth measure on $Gr_2(\mathbb{R}^{n+1})$. Theorem 7.1 implies that Φ is positive and unique. ☐

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