An Introduction to Finsler Geometry

BY Juan Carlos Álvarez and Carlos Durán

Preface

H. Busemann wrote as the first sentence of his classical survey on Finsler geometry that “ the term “Finsler space” evokes in most mathematicians the picture of an impenetrable forest whose entire vegetation consists of tensors”. He then went on to prove that the nearly exclusive use of tensors in the field was a historical accident.

In these lectures we will give a bare-bones introduction to Finsler geometry which shares, to a certain degree, Busemann's point of view. We started from the conviction that Finsler geometry lies at the intersection of convex geometry, metric geometry, and variational calculus, which are, and have been for sometime, in the mainstream of mathematics. It follows that Finsler geometry should also be a mainstream topic if an effort were made to present the basic definitions, theorems, and problems in a way which brings out its relation with these three fields.

The second conviction which served us as a guiding principle is that any point of view, any approach, and any theory is good if it produces theorems you would be happy to share with your colleagues at tea-time.

But what *is* a Finsler manifold? A quick intuitive definition is that it is a manifold which is infinitesimally a Banach space. By this we mean that on each tangent space we have a Banach norm which varies smoothly with the base point. Such a structure allows us to define the length of a smooth curve by integrating the Banach norm of the velocity vector at each point of the curve. We can then define the distance between two points as the infimum of the lenghts of all smooth curves joining them.

The functional that assigns to each smooth curve its length provides us with a variational problem which, if we want to study it, imposes some regularity conditions on the Banach norms defined on the different tangent spaces of the manifold.

Thus, we see that from the start Finsler geometry inherits something from convex geometry, metric geometry, and variational calculus. A nice theorem, or a nice problem, is one that brings out unexpected relations beween (at least) two of the three fields. An example of a 'nice' theorem is Akbar-Zadeh's result which states that a compact Finsler surface with constant negative curvature is Riemannian (see lecture 7). In these lectures we have tried to include as many nice theorems, problems, and examples as possible. We do not claim to be yet experts in the field, nor do we pretend to do justice to every single approach or result on the subject. Our ignorance makes this impossible, and the number of papers makes it excusable.

The lectures are naturally divided in two parts : lectures 1 through 4 treat the geometry of Minkowski spaces (finite dimensional Banach spaces with a strong smoothness condition) and their submanifolds while lectures 5 through 7 treat the intrinsic geometry of Finsler manifolds with a special emphasis on surfaces. Many of the problems and some of the theorems, specially in lectures 5 and 6, seem to be new.

This little book is dedicated to our wives, both present and future.

Acknowledgements

The authors thank all those that have shared their insight and their enthusiasm on the topics treated in these lectures. Specially useful for the first author have ben conversations with S. Tabachnikov, I.M. Gelfand, E. Lutwak, and E. Fernandes. We hope that those that helped us will enjoy these notes and that they will continue to press us for a better a more complete version which we will be sure to finish one day.

The first author is happy to thank TALVEN which made possible his participation in the *Escuela Venezolana de Matématicas.*

Our deepest expression of gratitude goes to the organizers of the *Escuela* who were kind enough to invite us many years ago as students and brave enough to invite us now as lecturers.

Apologies from the first author

This book should have been written in Spanish, but many years outside Venezuela have taken their toll on my ability to write coherently in the language of Cervantes. I hope that I will not be judged too severly and that the wider readership will compensate for my transgression.

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Lecture 1. Convex Geometry and Minkowski Spaces

In this lecture we study the geometry of Banach spaces under some smoothness assumptions as preparation for the study of Finsler manifolds.

1. Basic concepts

1.1 Definition. Let V be a finite dimensional real vector space and let $S \subset V$ be a hypersurface. We shall say that S is *convex* if its interior is a bounded convex subset of V. A convex hypersurface S is said to be *centrally symmetric* if whenever v is on S, $-v$ is also on S.

Note that we do not consider a small piece of a convex hypersurface to be a convex hypersurface. This may be contrary to common practice, but the alternative is to add the words 'closed' and 'bounded' to every ocurrence of the the words 'convex hypersurface' in the pages that follow.

Recall that the choice of a centrally symmetric convex hypersurface in a vector space is equivalent to the choice of a norm:

Let $S \subset V$ be a convex hypersurface which is closed and symmetric about the origin, let v be a nonzero vector in V, and let $\hat{v}$ to be the intersection of S with the ray from the origin to v. Define $\|v\|$ to be the unique real number satisfying $\|v\|\hat{v} = v$. If v is the zero vector we define $\|v\| = 0$. Thus, the hypersurface S defines a function $\|\cdot\|$ from V to the set of nonnegative real numbers, $\mathbf{R}^+$.

1.2 Proposition. *The function* $\|\cdot\| : V \to \mathbf{R}^+$ *is a norm.*

Proof. To show that $\|\cdot\|$ is a norm we have to prove that it satisfies the following conditions :
(1) $\|v\| \geq 0$ and equality holds if and only if $v = 0$.
(2) $\|\lambda v\| = |\lambda|\|v\|$ for all $v \in V$ and all $\lambda \in \mathbf{R}$.
(3) $\|v + w\| \leq \|v\| + \|w\|$ for all $v, w \in V$.

The first two are conditions are trivial to verify. To prove the third note that if v and w are two nonzero vectors in V, then the vectors $v/\|v\|$ and $w/\|w\|$ are on S and, by the convexity of S, the vector

$$\frac{\|v\|}{\|v\| + \|w\|}\frac{v}{\|v\|} + \frac{\|w\|}{\|v\| + \|w\|}\frac{w}{\|w\|} = \frac{v + w}{\|v\| + \|w\|}$$

is either on S or in the interior of S. This says that the norm of $(v + w)/(\|v\| + \|w\|)$ is less than or equal to one. It follows that $\|v + w\| \leq \|v\| + \|w\|$.

The remaining case, that of one or both vectors being zero, is trivial. ∎

1.1 Exercise. Show that if a function $\|\cdot\| : V \to \mathbf{R}^+$ is a *norm*, then the *unit sphere*, $\{v \in V : \|v\| = 1\}$ is a centrally symmetric convex hypersurface. Show also that the norm is a smooth function on $V \setminus 0$ if and only if the unit sphere is smooth.

Remark. For the rest of the lectures we shall deal only with smooth norms or, equivalently, with smooth convex hypersurfaces. We do this even when a definition or a result does not require smoothness in a fundamental way.

Two important concepts in the interface of convex geometry and the calculus of variations are the Legendre transform and duality.

1.3 Definition. Let V be a finite dimensional real vector space and let $S \subset V$ be a smooth convex hypersurface which is centrally symmetric. The *Legendre transform* associated to S is the map $\mathcal{L} : S \to V^*$ which takes a point v in S and sends it to the unique covector $\mathcal{L}(v)$ such that $\mathcal{L}(v)(w) = 1$ for any vector w on the tangent hyperplane to S at the point v. The image of S under the Legendre transform is the *dual hypersurface* to S and is denoted by S^*.

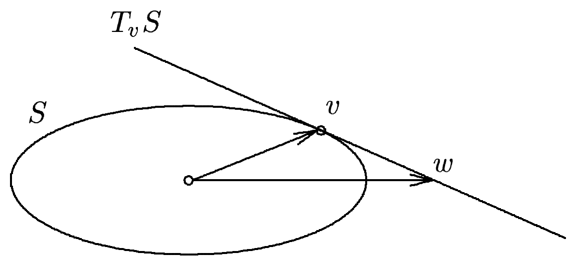

Fig. 1.1. Definition of the Legendre transform.

1.2 Exercise. Let V be a vector space, let $\varphi : V \to \mathbf{R}^+$ be a smooth norm function, and let S be the unit sphere $\{v \in V : \varphi(v) = 1\}$. Show that the map $d\varphi : S \to V^*$ is the Legendre transform.

Note that even if the unit sphere is smooth, the dual is not necessarily so. For example, if the unit sphere has a flat piece, the Legendrian transform will not be an embedding of S into V^*. Since we are doing differential geometry, and since we would like to treat the unit sphere and its dual as equals, we further restrict our class of normed spaces :

1.4 Definition. A centrally symmetric convex hypersurface S in a vector space V is said to be *quadratically convex* if it and its dual are smooth. A vector space V provided with a norm $\|\cdot\|$ such that the unit sphere is quadratically convex is called a *Minkowski space.*

In order to rewrite this definition in terms of the norm, and to justify the word 'quadratic' , let $L(\cdot) := \frac{1}{2}\|\cdot\|^2$, and define for each point v in the unit sphere the bilinear form $D^2L(v)$ as follows:

Let w_1 and w_2 be two vectors in V and let $\alpha(s,t)$ be a smooth vector valued function such that $\alpha(0,0) = v$, $\frac{\partial \alpha}{\partial s}(0,0) = w_1$, and $\frac{\partial \alpha}{\partial t}(0,0) = w_2$. Set

$$D^2L(v)(w_1, w_2) = \frac{\partial^2}{\partial s \partial t} L(\alpha(s,t))_{|(0,0)} \; .$$

1.3 Exercise. Show that

(1) $dL(v) \in V^*$ equals $\mathcal{L}(v)$.
(2) $D^2L(v)$ is well defined.
(3) $D^2L(v)(v,v) = 1$.
(4) The ellipsoid $\{w \in V : D^2L(v)(w,w) = 1\}$ is the unique elliposoid which is centered at the origin and which osculates S to second order at v.
(5) If w_1 and w_2 are tangent to S at a point v, then $D\mathcal{L}(v)(w_1) \cdot w_2 = D^2L(v)(w_1, w_2)$.

The following result follows easily from point (5) in the previous exercise :

1.5 Proposition. *A normed space $(V, \|\cdot\|)$ is a Minkowski space if and only if $D^2L(v)$ is a positive definite bilinear form.*

In great part our insistence on the smoothness of the unit sphere and its dual is that we want geodesics to be well-behaved. The following exercise makes this precise :

1.4 Exercise.

(1) Show that the shortest curve joining two points in a Minkowski plane is a line segment.

(2) Show that a curve of minimal length joining two points in the plane with the norm $\|(x,y)\| = \max\{|x|,|y|\}$ is not necessarily a line segment.

(3) Smooth out the norm in (2) to obtain a smooth norm with the same 'bad' property.

2. Isometries of Minkowski planes

In Euclidean geometry the group of rigid motions plays a fundamental role and intervenes in the introduction of concepts as well as in powerful techniques such as the method of moving frames. In Minkowski geometry the group of motions plays a modest role, nevertheless it is important to understand this role and to study the ways in which different Minkowski planes can be distiguished.

The first easy remark on the group of isometries of a Minkowski plane is that it contains all translations. The first natural question is whether it is contained in the group of affine transformations.

2.1 Proposition. *Any isometry of a Minkowski space is an affine transformation.*

Sketch of the proof. An isometry must map straight to straight lines and it clearly maps parallel lines to parallel lines. Any such transformation is necessarily affine. ∎

1.5 Exercise. Consider two Minkowski spaces with unit spheres S_1 and S_2. Show the the spaces are isometric if and only there exists an invertible linear transformation taking S_1 to S_2.

1.6 Exercise. Show that the group of invertible linear transformations that map a centrally symmetric convex curve onto itself is infinite if and only if the curve is an ellipse.

A more detailed study of the group of isometries is possible thanks to a beautiful construction due to Löwner:

Let S be a centrally symmetric convex hypersurface and let $< \cdot,\cdot >$ be a Euclidean inner product such that the Euclidean norm of any point on S is greater than or equal to 1. We now measure the volume of the region bounded by S with the volume form induced by the inner product $< \cdot,\cdot >$. Notice that $E := \{v \in V :< v,v >= 1\}$ contained in the region bounded by S and that the closer E is to S the smaller the volume.

2.2 Theorem-Definition (Löwner). *If S is a centrally symmetric convex hypersurface, then of all the Euclidean inner products $< \cdot,\cdot >$ with $E := \{v \in V :< v,v >= 1\}$ contained in S there is one and only one for which the volume of the region enclosed by S is minimal for the volume form induced by $<,\cdot,\cdot>$. This inner product is called the Löwner inner product.*

Idea of the proof of 2.2 . It is clear that there exists at least an inner product for which the area of S is minimal. To show that such an inner product is unique we argue by contradiction and assume there are at least two of them. Let us call them $< \cdot,\cdot >_1$ and $< \cdot,\cdot >_2$.

The key idea is to define the inner product $< \cdot,\cdot >_3 := \frac{1}{2}(< \cdot,\cdot >_1 + < \cdot,\cdot >_2)$, to notice that if v is a point on S, then $< v,v >_3 \geq 1$, and to prove that the volume of the region bounded by S for the volume form induced by $< \cdot,\cdot >_3$ is strictly smaller than for the volume forms induced by $< \cdot,\cdot >_1$ and $< \cdot,\cdot >_2$ unless these two inner products are identical. ∎

2.3 Corollary. *The group of isometries of a Minkowski space is a subgroup of the group of isometries of the euclidean espace defined by the Löwner inner product.*

Proof. From the definition of the Lowner inner product it is clear that if a linear transformation maps the unit sphere S onto itself, then it will map the Lowner inner product onto itself. ∎

1.7 Exercise. Show that the Lowner ellipse of a centrally symmetric convex curve touches the curve in at least four points.

3. The Invariant I

We now introduce an invariant, due to Cartan, which distinguishes two-dimensional Minkowski spaces and is of great importance in the Finsler geometry of surfaces. The idea is that a Minkowski plane defines a family of Euclidean planes parametrized by a projective line.

3.1 Proposition. *Let $S \subset \mathbf{R}^2$ be a centrally symmetric convex curve and let v be a point on S. There exists a unique ellipse $E(v)$ which is centered at the origin and osculates S up to second order at v.*

To define the invariant I we take a point v on a centrally symmetric convex curve S and complete v to an oriented orthonormal basis, $\{v, Jv\}$, for the Euclidean structure define by the ellipse $E(v)$ (see figure 2).

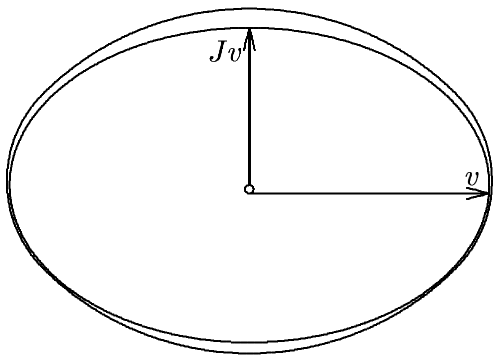

Fig. 1.2. The definition of Jv.

1.8 Exercise. Think about the following concept due to Cartan:

3.2 Definition. Two vectors w_1 and w_2 based respectively at points v_1 and v_2 of the curve S are said to be *equipolent* if the coordinates of w_1 with respect to the basis $\{v_1, Jv_1\}$ are equal to those of w_2 with respect to the basis $\{v_2, Jv_2\}$.

Let us parametrize the curve S by a map γ such that $\dot{\gamma}(t) = J\gamma(t)$.

3.3 Definition. The period of the curve γ will be called the *total angle* of S.

1.9 Exercise. Prove that the total angle of two linearly equivalent centrally symmetric convex curves is the same.

3.4 Proposition-Definition. *The components of $\ddot{\gamma}(t)$ in the basis $\{\gamma(t), \dot{\gamma}(t)\}$ depend only on the curve S and the point v. The invariant $\mathbf{I}(t)$ is the component of $\ddot{\gamma}(t)$ in the direction of $\dot{\gamma}(t)$.*

1.10 Exercise. Prove the previous proposition.

The function $\mathbf{I}$ is a linear invariant of the curve S in the sense that if A is an invertible linear transformation from the plane to itself, then the value of the $\mathbf{I}$ invariant of the curve $A(S)$ at a point v equals the value of the $\mathbf{I}$ invariant of the curve S at the point $A^{-1}(v)$.

1.11 Exercise. Prove that $\mathbf{I}(t)$ equals zero if and only if the ellipse $E(\gamma(t))$ osculates the curve S to third order at the point $\gamma(t)$. Deduce that the function $\mathbf{I}$ is identically zero if and only if S is an ellipse.

1.12 Exercise. Let $F : \mathbf{R}^2 \to \mathbf{R}$ denote the square of the norm defined by the centrally symmetric convex curve S. Assuming that S is parametrized as above, show that the $\mathbf{I}$ invariant of S has the following expression in terms of F:

$$\mathbf{I}(v) := \frac{1}{2} \cdot \frac{d}{dt} \ln \begin{vmatrix} \frac{\partial^2 F}{\partial x^2} & \frac{\partial^2 F}{\partial x \partial y} \\ \frac{\partial^2 F}{\partial x \partial y} & \frac{\partial^2 F}{\partial y^2} \end{vmatrix},$$

where the Hessian of F is evaluated at the point $\gamma(t)$.

Here is an interesting global result about the invariant $\mathbf{I}$:

3.5 Theorem. *For any centrally symmetric convex curve S, the invariant $\mathbf{I}$ vanishes at least eight times.*

Proof. Consider the transformation $P : \mathbf{RP}^1 \to \mathbf{RP}^1$ which sends a line l passing through the origin to the line passing through the origin and parallel to the lines tangent to S at the the points of intersection of l and S (see figure 1.3).

1.13 Exercise. Show that the transformation P is projective if and only if S is an ellipse.

The transformation P is approximated to third order by a projective transformation at a line l if and only if the osculating ellipse at the points of intersection of l and S osculates to third order. Exercise 1.7 tells us that this happens if and only if the invariant $\mathbf{I}$ vanishes at the points of intersection of l and S.

The *coup de grâce* is given by a theorem of Ghys (see [OT]) which states that any diffeomorphism of the projective line has at least four points at which it is approximated to third order by a projective transformation. ∎

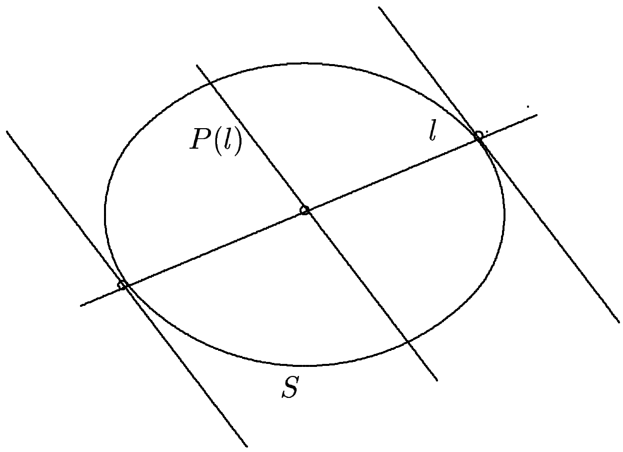

Fig. 1.3. Definition of P.

To conclude the lecture we would like to state a problem that seems to be open and which has interesting applications in Finsler geometry. We state it in the simplest possible case. The reader is allowed to generalize it, provided he solves it.

Problem. *Is there any centrally symmetric convex surface in $\mathbf{R}^3$ which is not an ellipsoid and such that for every plane Π passing through the origin the total angle of $\Pi \cap S$ is the same.*

References

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[TH] Thompson, A.C., *Minkowski geometry* . Cambridge Univ. Press, 1996.

[OT] Ovsienko, V. and Tabachnikov, S., Sturm theory, Ghys theorem on zeroes of the Schwarzian derivative and flattening of Legendrian curves, *Selecta Math., New Series, v. 2* **2** (1996), 297-307.

Lecture 2. Length and Area

After having studied the basics of Minkowski spaces we now take the study of the geometry of curves in Minkowski planes. Our official excuse for doing this in a minicourse on Finsler geometry is the following:

When young students take a first course on Riemannian geometry they go in armed with all the intuition, motivation, and good taste provided by elementary Euclidean geometry, the theory of plane curves, and the theory of space curves and surfaces. This intuition is badly missed by the student of Finsler geometry and we would like to remedy this to a small extent. Luckily, there is now an excellent text [Th] to complement this lecture.

The unofficial reason is that the theory is as pretty and classical as in the Euclidean case and provides us with a unique oportunity to relearn from a new viewpoint those results that once attracted us to geometry.

1. Length

Let us start with the simplest and most natural concept in Minkowski geometry:

1.1 Definition. Let $\gamma : [a, b] \to \mathbf{R}^2$ be a regular parametrized curve in the Minkowski plane $(\mathbf{R}^2, \|\cdot\|)$. The *length* of γ is the integral $\int_a^b \|\dot{\gamma}(t)\| dt$.

2.1 Exercise. Show that the length of a curve does not depend on its parametrization.

Even with such a simple concept there is already some nontrivial work to do. For example, consider the number π. In Euclidean geometry π is one-half the length of the unit circle. Is it possible that for some strange being living in a Minkowski plane the number "π" be equal to 4? The answer, as given by the following theorem of Golab, is no.

1.2 Theorem ([Go]). *The length of a unit circle in a Minkowski plane is greater than 6 and less than 8.*

To give an idea of how easy-looking things can be nontrivial we encourage to try the following problem before we give the answer later in the lecture.

2.2 Exercise. If C_1 and C_2 are two smooth convex curves with C_2 inside C_1, which of the two curves is longer?

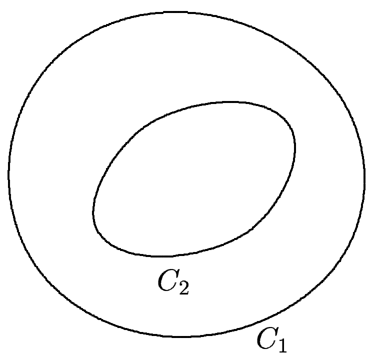

Fig. 2.1. Which is longer? Obviously ... .

The proof of Golab's theorem is very easy once one has proved that the curve inside is shorter.

2.3 Exercise. Deduce from the following figure that one can always inscribe a regular hexagon in the unit circle of a Minkowski plane. Note that the length of each side equals one and so the perimeter equals six.

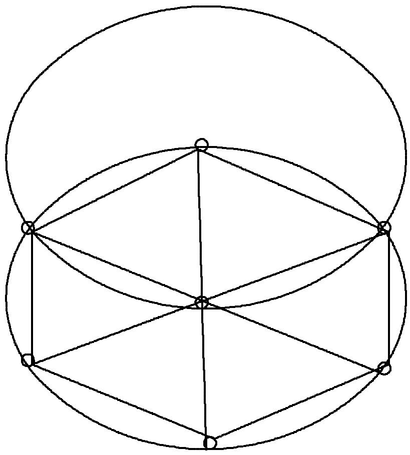

Fig. 2.2. An inscribed regular hexagon.

2.4 Exercise. Deduce from the following figure that one can always inscribe the unit circle of a Minkowski space in a parellelogram. Note that each side has length two and so the perimeter equals eight.

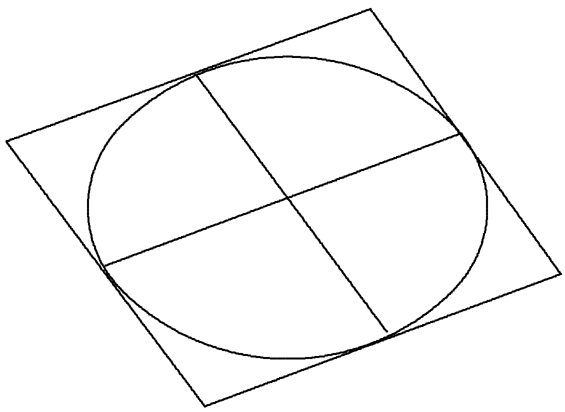

Fig. 2.3. An inscribing parallelogram.

Here is another nice result regarding the length of the unit circle :

1.3 Theorem. *The length of the unit circle of a Minkowski plane equals the the length of its dual.*

We do not know who was the first to discover and prove this theorem, but it can be found in [T]. In lecture 6 we shall give its proof as a very particular case of a theorem in symplectic geometry.

2. Area

Before coming back to the problem of the nested convex curves we introduce the concept of area in a Minkowski plane and in order to do this we introduce just a spoonful of symplectic geometry.

2.1 Definition. The symplectic form ω_0 on $\mathbf{R}^2 \times \mathbf{R}^{2*}$ is the bilinear form defined by

$$\omega_0((v_1,\xi_1),(v_2,\xi_2)) := \xi_1(v_2) - \xi_2(v_1) \ .$$

The volume form Ω_0 on $\mathbf{R}^2 \times \mathbf{R}^{2*}$ equals $\omega_0 \wedge \omega_0$.

2.2 Definition ([HT]). Let $U \subset \mathbf{R}^2$ be an open set and let $D^* \subset \mathbf{R}^{2*}$ be the region bounded by the dual unit sphere S^*. We define the *Holmes-Thompson area* of U to be the symplectic volume of $U \times D^*$ divided by π.

This definition coincides with the standard notion of area if the plane is Euclidean.

2.5 Exercise. Show that the Holmes-Thompson area of a set is invariant under translations.

From this point on, we will not use any other definition of area so that the word 'area' will stand for 'Holmes-Thompson area' .

As a consequence of this exercise we have that the definition of area agrees with the Euclidean up to a multiple which can be obtained by measuring the area of the unit disc (yet another relative of π!).

Theorem (Blaschke). *The area of the unit disc of a Minkowski plane is less than or equal to π. Moreover equality holds if and only if the plane is Euclidean.*

A simple proof of this theorem and of its generalization to higher dimensions, the Blaschke-Santalo inequality, is given in [MP].

Problem. *Are there other Minkowski planes besides the Euclidean for which the ratio of the length of the unit circle to the area of the unit disc equals 2 ?*

3. Integral geometry

In Euclidean geometry one can use the Euclidean group not only to define an invariant measure in the plane, but also to define an invariant measure on the set of all oriented lines on the plane. Using this measure on the space of lines we have *Crofton's theorem* which states that the length of a curve equals the measure of the set of lines which intersect it, provided we count each line as many times as it intersects the curve.

In this section we shall show that the set of all oriented lines of a Minkowski plane has a natural measure which makes possible the extension of Crofton's theorem. This was first shown by Blaschke for the more general setup of Finsler surfaces in [Ba]. Here we follow the simpler symplectic treatment from [Al].

Let us start by identifying the set of oriented lines in a Minkowski plane with the tangent bundle of its unit circle by the following geometric construction :

Given an oriented line l consider the unit vector v passing through the origin and with the same direction oas l. The tangent line T_vS intersects l at a point. We identify l with this point in T_vS.

The construction shows that the set of oriented lines is a Minkowski plane is a manifold diffeomorphic to the cylinder. We shall denote it by $H_1(\mathbf{R}^2)$.

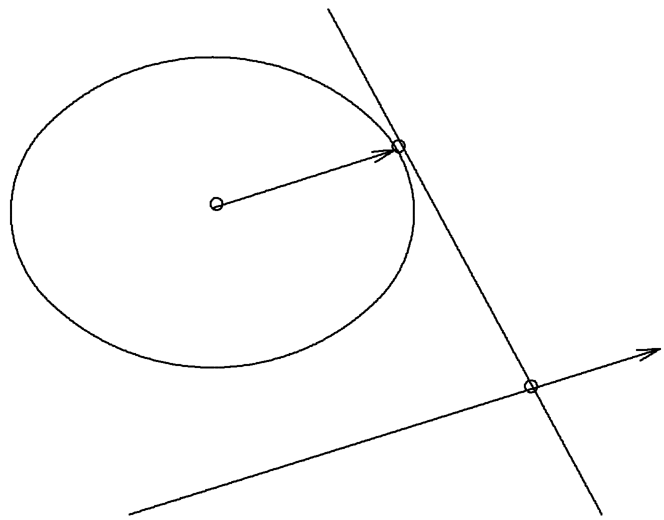

Fig. 2.4. Identification of $H_1(\mathbf{R}^2)$ and TS.

In order to define the area form on the set of oriented lines we consider the inclusion i : $\mathbf{R}^2 \times S^* \to \mathbf{R}^2 \times \mathbf{R}^{2*}$ and the projection $\pi : \mathbf{R}^2 \times S^* \to H_1(\mathbf{R}^2)$ which maps a pair (v, ξ) to the line passing through v in the direction of $\mathcal{L}(\xi)$, where $\mathcal{L} : S^* \to S$ is the Legendre transform.

3.1 Definition. The area form ω on the space of lines of a Minkowski plane is defined by the equality $\pi^*\omega = i^*\omega_0$.

Encoded in this slick definition is that in order to evaluate ω on a pair of tangent vectors in the space of lines we must first choose two tangent vectors in $\mathbf{R}^2 \times S^*$ which project onto them and then compute the value of ω_0 on these vectors.

2.6 Exercise. Show that ω is well defined.

Theorem [Bl]. *If $\gamma : [a,b] \to \mathbf{R}^2$ is a smooth immersed curve, then*

$$\int_a^b \|\dot{\gamma}(t)\| \, dt = \frac{1}{4} \int_{l \in H_1(\mathbf{R}^2)} \#(\gamma([a,b]) \cap l) \ \omega \ .$$

Before going into the proof let us show how this theorem helps us solve exercise 2.2.

3.2 Proposition. *If C_1 and C_2 are two smooth convex curves with C_2 inside C_1, then C_1 is strictly longer than C_2.*

Proof. By the previous theorem, the length of a closed convex curve equals one-half the area of the set of oriented lines passing through it. Since any line passing thorugh C_1 passes through C_2 we have that C_1 is at least as long as C_2. To have the strict inequality we merely note the set of oriented lines which pass through C_1, but not through C_2 contains an open set. ∎

Before attempting a proof of the Crofton formula above, let us notice that it is sufficient to prove it in the case where the curve γ is a line segment.

2.7 Exercise. Show that the Crofton formula for line segments implies the Crofton formula for polygonal curves and use this to deduce the general Crofton formula.

So, all that remains for us is to prove the following

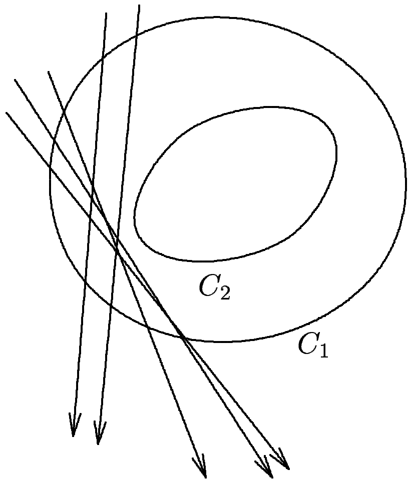

Fig. 2.5. C_1 is longer than C_2.

3.3 Lemma. *The distance between two points in a Minkowski plane equals the one-fourth the symplectic area of the set of oriented lines passing through the segment that joins them.*

To elucidate the relation between the symplectic form ω and the element of arclength let us first consider $\mathbf{R}^2 \times \mathbf{R}^{2*}$ as the cotangent of $\mathbf{R}^2$ and the form ω_0 as a differential form on $T^*\mathbf{R}^2$. Note that if (q_1, q_2, p_1, p_2) are the canonical coordinates of $T^*\mathbf{R}^2$, then $\omega_0 = dp_1 \wedge dq_1 + dp_2 \wedge dq_2$ and hence ω_0 is the exterior differential of the canonical 1-form $\alpha_0 := p_1 dq_1 + p_2 dq_2$. For future reference we remark α_0 vanishes identically on any curve consisting of covectors based at given point; the quantities dq_1 and dq_2 are zero in this case.

It is the canonical 1-form that is directly related to the element of arclength :

2.8 Exercise. Show that the length of the segment $\{m + tv : t \in [0,1]\}$ equals the integral of α_0 along the curve $t \mapsto (m + tv, \mathcal{L}(v))$, $t \in [0,1]$.

Proof of the lemma. Let m and n be two points on the plane and let $\hat{m}$ and $\hat{n}$ denote the pencils of oriented lines passing through m and n respectively. In the manifold of oriented lines $\hat{m}$ and $\hat{n}$ are two circles which intersect at two points, the two oriented lines joining m and n, and which cut $H_1(\mathbf{R}^2)$ into four open connected components, two of them bounded, two unbounded. The union of the two bounded components is the set of all oriented lines passing through the open segment from m to n. We aim to show that the symplectic area of each of its connected components equals twice the distance between m and n.

Let's take one of this bounded components, denote it by U_1, and consider its boundary. This boundary is made up of pieces of $\hat{m}$ and $\hat{n}$ as shown in figure 2.6.

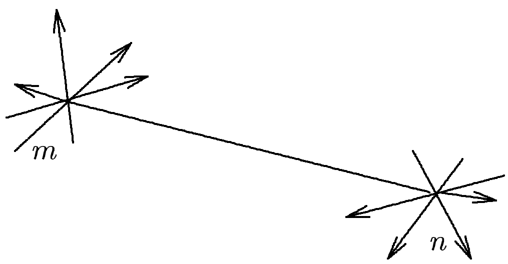

Fig. 2.6.

Note that the boundary of U_1 can seen as the image of a *closed* piecewise smooth curve $\Gamma \subset \mathbf{R}^2 \times S^*$ under the projection $\pi : \mathbf{R}^2 \times S^* \to H_1(\mathbf{R}^2)$ which sends a pair (v, ξ) to the line passing through v in the direction of $\mathcal{L}(\xi)$. Indeed, $\pi^{-1}(\partial U_1)$ is made up of two disjoint curves consisting of unit covectors over the points m and n, respectively. We form Γ by joining these two curves with the curves

$$c_1 := \{(m + t(n-m), \mathcal{L}(n-m)) : t \in [0,1]\} \text{ and}$$
$$c_2 := \{(n + t(m-n), \mathcal{L}(m-n)) : t \in [0,1]\} .$$

The curve Γ is the boundary of a disc D in $\mathbf{R}^2 \times S^*$ such that $\pi(D) = U_1$ and hence

$$\int_{U_1} \omega = \int_{\pi(D)} \omega = \int_D \pi^* \omega = \int_D \omega_0 = \int_\Gamma \alpha_0 .$$

It remains for us to prove that the last integral equals twice the distance from m to n. Fortunately this involves almost no work: the form α_0 on the pieces consisting of unit covectors over the points m and n is zero and, by exercise 2.9, the integral of α_0 along either c_1 or c_2 equals the distance between m and n. ∎

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Lecture 3. Normals in Minkowski Spaces

In this lecture we continue reviewing elementary concepts in Minkowski geometry. We turn to the question of normality in Minkowski spaces and, specially, in Minkowski planes. The main references for this lecture are the book [Th] and the paper [Ta].

1. Normals in Minkowski planes

In Euclidean geometry the normal line to a smooth curve satisfies two properties :

(1) If we parametrize the curve by arclength, the the acceleration vector lies on the normal.

(2) If n is any point on the line which normal to the curve at a point m, then m is a critical point of the function $x \mapsto \|x - n\|$, where x ranges over the points on the curve.

Both ideas can be applied to to give a definition of normal in Minkowski geometry, but they do not coincide. We prefer to give simple synthetic definitions for both types of normals and let the reader see that they respectively satisfy (1) and (2).

1.1 Definition. If $l \subset \mathbf{R}^2$ is a line through the origin, then its *acceleration normal* is the line passing through the origin and parallel to the lines tangent to S at the points of intersection of l and S.

1.2 Definition. If $l \subset \mathbf{R}^2$ is a line through the origin, then its *Minkowski normal* is the line passing through the origin and intersecting S at precisely those points where the tangent line is parallel to l.

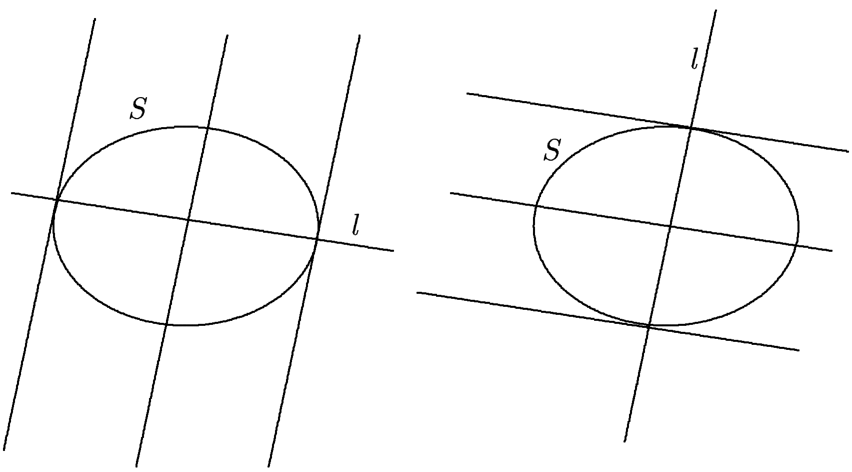

Fig. 3.1. Acceleration and Minkowski normals.

3.1 Exercise. If γ is a smooth curve parametrized by arc-length, then for each value of the parameter t the vector $\ddot{\gamma}(t)$ lies in the acceleration normal to $\dot{\gamma}(t)$.

3.2 Exercise. Let γ be a smooth regular curve and let m be a point on γ. If n is a point on the Minkowski normal of γ at the point m, then m is a critical point of the function $x \mapsto \|x - n\|$, where x ranges through the points on γ.

3.3 Exercise. Characterize all Minkowski planes for which these two concepts of normality coincide.

The two concepts of normality are closely related. Namely, for any Minkowski plane with unit circle S there exists an associated Minkowski plane with unit circle $\hat{S}$ such that the acceleration normal of any line in the first Minkowski plane equals its Minkowski normal in the second.

1.3 Proposition. *Let S be a convex centrally symmetric curve and let $\gamma(t)$ be the parametrization of S satisfying $\det(\dot\gamma(t), \gamma(t)) = 1$. The curve $\dot\gamma(t)$ parametrizes a convex centrally symmetric curve $\hat{S}$ such that the acceleration normal of any line with respect to S equals its osculating normal with respect to $\hat{S}$.*

Proof. All that has to be shown is that the tangent line to $\hat{S}$ at $\dot\gamma(t)$ is parallel to $\gamma(t)$ or, equivalently, that $\ddot\gamma(t)$ is a multiple of $\gamma(t)$. To see this we differentiate the equation $\det(\dot\gamma(t), \gamma(t)) = 1$ with repect to t and obtain $\det(\ddot\gamma(t), \gamma(t)) = 0$. ∎

Many global theorems in classical geometry depend on the properties of normals. For example, the four-vertex theorem depends greatly on the fact that the Euclidean normals of a closed curve cover the whole plane. This property remains true for normals in Minkowski geometry, no matter which type of normals we take.

1.4 Proposition. *Let σ be a closed immersed curve in a Minkowski plane. If $p \subset \mathbf{R}^2$ is any point, then there exist at least one line passing through p and which is an acceleration (Minkowski) normal of σ.*

Proof. The proof in the case of Minkowski normals is quite easy: the function $x \mapsto \|x - p\|$, where x rages over all points $x \in \sigma$, has at least two critical points. The lines passing through p and through these two points are Minkowski normals.

In the case of acceleration normals we use the previous proposition and reduce the proof the the case of Minkowski normals. ∎

In the figure that follows, the family of lines tangent to the small circle is transversal to the curve σ, but it is neither the family of Minkowski nor acceleration normals of σ in any Minkowski plane.

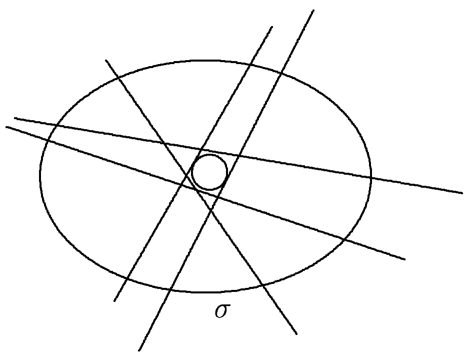

Fig. 3.2. A family of tranversals which are not normals.

1.5 Definition. The set of all points at a distance $r > 0$ from a point p is called a *Minkowski circle* with center p and radius r.

3.4 Exercise. Show that Minkowski circles satisfy the following properties:

- If two Minkowski circles intersect in three or more points, then they are equal
- If two Minkowski circles are tangent up to order two at any point, then they are equal.
- If γ is any smooth immersed curve, then at any point $\gamma(t)$ there exists a unique Minkowski circle that osculates γ to order two.

1.6 Definition. Let $\gamma \subset \mathbf{R}^2$ be a smooth immersed curve. The Minkowski curvature of γ at a point $\gamma(t)$ is the radius of the unique Minkowski circle which osculates γ to order two at $\gamma(t)$.

Perhaps the first global theorem in differential geometry which one learns as an undergraduate student is the *four-vertex theorem* . We shall show that this theorem can be extended to Minkowski geometry and sketch a proof using the following result of Segre, recently rediscovered (and revived) by Arnold.

1.7 Theorem ([Se] and [Ar]). *A smooth embedded curve on the (euclidean) unit sphere whose convex hull contains the origin has at least four inflection points.*

Other approaches and a much more detailed account of the four-vertex theorem in Minkowski geometry can be found in [Um] and [Ta].

1.8 Theorem. *Let γ be a smooth convex curve in a Minkowski plane. The Minkowski curvature function has at least four critical points.*

Sketch of proof. We start by identifying the set of oriented lines in the plane with the cylinder $\{(x, y, z) \in \mathbf{R}^3 : x^2 + y^2 = 1\}$:

Consider the plane as the plane $z = 1$ inside $\mathbf{R}^3$. For each oriented line l on $z = 1$ take the Π which contains the line and passes through the origin. We coorient Π by prescribing that its positive side be given by the cross product of any vector from the origin to the line times any vector in the direction of the line.

The line which passes through the origin and is perpendicular to this plane intersects the cylinder in two points. The point which we assign to l is that lying on the positive side of Π.

3.5 Exercise. Show that the pencil of lines passing through a given point is identified with the intersection of the cylinder with a plane passing through the origin.

The preceeding identification allows us to view the set of all Minkowski normals of γ as a curve on the cylinder. We shall call this curve the *normal system* of γ.

3.6 Exercise. The normal system is a smooth curve and its inflection points are in one-to-one correspondence with the critical points of the Minkowski curvature function.

To prove the four-vertex theorem we have to show that the normal system has at least four inflection points. It is here where we see the relation with theorem 1.7.

Lemma. *The convex hull in $\mathbf{R}^3$ of a normal system contains the origin.*

Proof of the lemma. We have seen that given any point p in the plane there is at least one Minkowski normal of γ that passes through p. By exercise 3.5, this is equivalent to saying that every plane through the origin cuts the normal system in at least a point. This proves that the convex hull contains the origin.

Note that the radial projection from the cylinder to the sphere takes curves whose convex hull contain the origin to curves with the same property. The radial projection also preserves the number of inflexion points on a curve.

We finish the proof by applying theorem 1.7 to the image of the normal system under the radial projection. ∎

2. Normals in higher dimensions

2.1 Definition. Let $(V, \|\cdot\|)$ be a Minkowski space with unit sphere S. If $\zeta \subset V$ is a hyperplane through the origin, then its *Minkowski normal* is the line passing through the origin and intersecting S at precisely those points where the tangent space is parallel to ζ.

The relation with critical point theory is the same as in the two-dimensional case : if $\Sigma \subset V$ is a hypersurface and if n is a point on the Minkowski normal of Σ at the point $m \in \Sigma$, then m is a critical point of the function $x \mapsto \|x - n\|$, where x ranges through the points on Σ.

Most results and problems in submanifold geometry which come from studying the critical points of the distance, or the distance squared, functions make sense in the Minkowski setting and can be interpreted geometrically by using the Minkowski normal. For example, here are two equivalent definitions of a *focal point*.

2.2 Definition. Let $(V, \|\cdot\|)$ be a Minkowski space and let $\Sigma \subset V$ be a smooth surface. A point $v \in V$ is said to be a *focal point* of Σ if the distance squared function $x \mapsto \|x - v\|^2$ is not a Morse function on Σ.

2.3 Definition. Let $(V, \|\cdot\|)$ be a Minkowski space and let $\Sigma \subset V$ be a smooth submanifold. Define the *normal bundle* of Σ, $\nu(\Sigma)$, as the set of pairs $(x, v) \in \Sigma \times V$ such that the line xv is the Minkowski normal of some hyperplane containing $T_x\Sigma$. A point is said to be *focal* if it is a singular point of the map $(x, v) \mapsto v$ from $\nu(\Sigma)$ to V.

The last definition may seem a bit complicated at first, but it just says that a point is focal is the Minkowski normals focus in at this point.

2.4 Definition. A compact submanifold M of a Minkowski space $(V, \|\cdot\|)$ is said to be taut if for almost every point $v \in V$, the distance squared function $x \mapsto \|x - v\|^2$ is a perfect Morse function on M.

2.5 Theorem ([AM]). *Let $(V, \|\cdot\|)$ be a Minkowski space and let $\Sigma \subset V$ be a compact submanifold which is homeomorphic to a sphere. If Σ is taut, then it is a Minkowski sphere.*

Here are some basic problems related to the theory of taut immersions in Minkowski 3-spaces.

Problem. *Characterize all taut surfaces in a three-dimensional Minkowski space.*

Problem. *Characterize all surfaces in a three-dimensional Minkowski space for which the set of focal points has dimension 1. These surfaces should be called Minkowski-Dupin surfaces.*

In the Euclidean case it can be proved that a Dupin surface is a torus and that its focal set consists of the union of two conics. Is it at least true that the focal set of a Minkowski-Dupin surface consists of pieces of planar curves?

We end the lecture by defining the analogue of acceleration normal in higher dimensions.

2.6 Definition. Let $(V, \|\cdot\|)$ be a Minkowski space with unit sphere S, let $\zeta \subset V$ be a hyperplane passing through the origin, and let v be a point $S \cap \zeta$. The v-normal of ζ is the is orthogonal to ζ with respect to the bilinear form $D^2L(v)$.

3.7 Exercise. Characterize all Minkowski spaces for which the v-normals of a two-dimensional subspace all coincide.

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Lecture 4. Submanifolds in Minkowski Spaces

A submanifold M in a Minkowski space $(V, \|\cdot\|)$ inherits a (Finsler) metric by the following simple procedure:

If x and y are points on M, define their distance as the infimum of the lengths of all smooth curves on M which join x and y, where the length of a smooth curve $\gamma : [a,b] \to V$ is given by the integral $\int_a^b \|\dot{\gamma}(t)\| dt$.

In this lecture we briefly study the geometry of submanifolds in a Minkowski space. We concentrate mostly on three-dimensional case since our main objective is to motivate the intrinsic Finsler geometry of surfaces which will be studied in the following three lectures.

1. Two characterizations of Euclidean 3-space

Convex geometry is full of delightful problems and theorems, most of which have (or should have) applications to Finsler geometry. Suppose, for example, that we want to know if there is a Minkowski space besides the Euclidean such that every k-dimensional submanifold inherits a Riemannian metric. It is then useful to have the following theorem :

1.1 Theorem. *Let V be a vector space and let $S \subset V$ be a centrally symmetric hypersurface. If every k-dimensional subsspace of V intersects S in an ellipsoid, then S is an ellipsoid.*

We shall give a very quick proof of the a particular case. For the general case, the reader is referred to page 91 of [Bu].

1.2 Proposition. *Let $S \subset \mathbf{R}^3$ be a convex centrally symmetric surface. If the intersection of S with any plane through the origin is an ellipse, then S is an ellipsoid.*

Proof. Let x and $-x$ be two points on S such that their distance, with respect to the standard Euclidean metric, equals the diameter of S. By hypothesis, the intersection of S with any plane containing these two points is an ellipse. Notice that the diameter of all these ellipses also equals the distance between x and $-x$ and therefore the segment between these two points is the major axis of all of them. The minor axes of these ellipses are then all on the plane Π which passes through the origin and is orthogonal to the line joining x and $-x$. The intersection of Π and S is again an ellipse and it is easy to see that the ellipsoid which contains this ellipse and passes through x and $-x$ coincides with S. ∎

Another simple question is whether we can find Minkowski spaces, other then the Euclidean, in which all k-dimensional subspace are isometric. This was asked by Banach who conjectured that the answer is no. Gromov settled many cases of the conjecture by using the topology of fiber bundle (see [Gr]).

Again, we shall give a quick proof of a very particular case.

1.3 Theorem. *If all 2-dimensional subspaces of a Minkowski 3-space are isometric, then the space is Euclidean.*

It is convenient to rewrite the theorem in its geometric form :

1.4 Theorem. *Let $S \subset \mathbf{R}^3$ be a convex centrally symmetric surface. If any two intersections of S with planes through the origin are linearly equivalent, then S is an ellipsoid.*

Proof. Since the case where all the intersections are ellipses was treated in proposition 1.2, we shall assume that the intersections are not ellipses.

The idea of the proof is to argue by contradiction and to show that a convex centrally symmetric surface which is not an ellipsoid, but which satisfies the hypothesis of the theorem can be used to construct a nowhere zero vector field on the sphere.

For a point x on the (euclidean) unit sphere, denote by $S(x)$ the intersection of S with the 2-dimensional subspace orthogonal to x. We think of $S(x)$ as a subset of T_xS^2.

We denote by $E(\mathbf{R}^2, TS^2)$ the vector bundle over S^2 whose fiber over a point x is the space of linear maps from $\mathbf{R}^2$ to T_xS^2, and we let C be a centrally symmetric ocnvex curve on $\mathbf{R}^2$ which is linearly equivalent to anyone of the $S(x)$.

Claim. The subset $E_S \subset E(\mathbf{R}^2, TS^2)$ is a smooth subbundle and the natural projection $E_S \to S^2$ is a covering map.

The claim follows simply from the following three facts :

(1) $S(x)$ changes smoothly with x
(2) The set of linear transformations taking C to $S(x)$ is finite for every x.
(3) For any x and y in S^2, $S(x)$ and $S(y)$ are linearly equivalent.

Note now that a covering map over the sphere always has a smooth section. In this case a section is the smooth shoice of a linear map $A_x : \mathbf{R}^2 \to T_xS^2$ for every $x \in S^2$. This would imply that the tangent bundle of the sphere is trivial and we arrive at our desired contradiction. ∎

Theorem 1.4 will be used in lecture 6 to show that certain kinds of Finsler surfaces admit no isometric embedding into a Minkowski 3-space.

2. Holmes-Thompson Area

There are two basic measurements that can be made in a Minkowski space the length of a curve and the k-area of a k-dimensional submanifold.

There are several natural, but inequivalent proposal to define the area of a k-dimensional submanifold of a Minkowski space. In this notes we follow the Holmes-Thompson definition because its relation with symplectic geometry ties it to the calculus of variations which lies at the heart of Finsler geometry.

2.1 Definition. *The canonical symplectic form* on $T^*V = V \times V^*$ is the skew-symmetric bilinear form

$$\omega_0((v_1, \xi_1), (v_2, \xi_2)) := \xi_1(v_2) - \xi_2(v_1) \ .$$

If $M \subset V$ is an embedded k-dimensional submanifold, then its cotangent bundle T^*M is naturally embedded in $V \times V^*$. We define the volume form on T^*M as the restriction of ω_0^k to T^*M.

Let us now define the set

$$D^*M := \{(m, \xi) \in T^*M \subset V \times V^* : \xi \text{ is in the interior of } S^*\} \ .$$

2.2 Definition. Let $(V, \|\cdot\|)$ be a Minkowski space with unit sphere S. The *Holmes-Thompson area* of a k-dimensional submanifold $M \subset V$, denoted by $A_{HT}(M)$, equals the integral of ω_0^k over D^*M divided by (euclidean) volume of the euclidean ball of dimension k.

4.1 Exercise.
(1) Show that the Holmes-Thompson area of a curve equals its length.
(2) Show that if $(V, \|\cdot\|)$ is Euclidean, then the Holmes-Thompson area equals the standard area.

Using the Holmes-Thompson area it is possible to generalize the fact that the length of the unit circle in a Minkowski plane equals that of its dual.

2.3 Theorem ([HT]). *The Holmes-Thompson area of the unit sphere in a Minkowski space equals that of its dual.*

The following problem is posed in [Th]:

Problem. *Are there sharp bounds on the area of the unit sphere of a Minkowski space? In other words, is there a Golab's theorem in higher dimensions?*

3. Geodesics on surfaces

Just as in the classical theorem of surfaces we define a *geodesic* to be a curve that locally minimizes length.

4.2 Exercise. Let $M \subset \mathbf{R}^3$ be an embedded surface and let $\gamma : [a,b] \to M$ be a smooth curve parametrized by arc-length. Show that γ is a geodesic if and only if it is an extremal of the action functional

$$A(\sigma) := \int_a^b \frac{1}{2}\|\dot\sigma(t)\|^2 \; dt \ ,$$

where σ rages over all smooth curves $\sigma : [a,b] \to M$ with $\sigma(a) = \gamma(a)$ and $\sigma(b) = \gamma(b)$.

Let $F : \mathbf{R}^3 \to \mathbf{R}$ be a smooth function such that 0 is a regular value of F and $F^{-1}(0) = M$. It is well known in the calculus of variations that a smooth curve γ on M is an extremal of the functional A if and only if it is an extremal of the functional

$$A_F(\sigma) := \int_a^b \frac{1}{2}\|\dot\sigma(t)\|^2 - \lambda F(\gamma(t) \; dt \ ,$$

where σ rages over all smooth curves $\sigma : [a,b] \to \mathbf{R}^3$ with $\sigma(a) = \gamma(a)$ and $\sigma(b) = \gamma(b)$.

We shall use this variation on the theme of Lagrange multipliers to prove the following basic result.

3.1 Proposition. *Let M be a surface in a Minkowski space and let $q : [a,b] \to M$ be a geodesic parametrized by arclength. For every value of the parameter t, the vector $\ddot q(t)$ lies on the $\dot q(t)$-normal to $T_{q(t)}M$.*

Proof. Let us denote $\frac{1}{2}\|\dot q(t)\|^2$ by $L(\dot q(t))$ and write the Euler-Lagrange equations

$$\frac{d}{dt} dL(\dot q) = \lambda dF(q) \ .$$

Since $\frac{d}{dt} dL(\dot q) = D^2L(\dot q)(\ddot q, \cdot)$, we have that if w is a vector on $T_{q(t)}M$, $D^2L(\dot q)(\ddot q, w) = \lambda dF(q)\cdot w = 0$ ∎

We now introduce an interesting metric invariant of embedded spheres in a Minkowski 3-space $(V, \|\cdot\|)$. This invariant was first considered by Birkhoff (see [Bi]) in his proof of the existence of one closed geodesic for any Riemannian metric on the sphere.

Let $M \subset V$ be an embedded sphere and let $f : M \to \mathbf{R}$ be a smooth function with only two critical points. Define

$$\beta(f) := \max\{ \text{ length of } f^{-1}(c) : c \in \mathbf{R}\} \ .$$

and set $\beta(M)$ to be the infimum of $\beta(f)$ as f rages over all smooth functions on M with only two critical points. Clearly β is a metric invariant of M. It's importance comes from the following theorem :

3.2 Theorem (Birkhoff). *There exists a closed geodesic on M with length $\beta(M)$.*

3.3 Proposition. *If M is the unit sphere of a Minkowski space, then $\beta(M) < 8$.*

Proof. Let $h; M \to \mathbf{R}$ be a height function. The level sets of h are the intersections of M with a family of parallel planes. The longer of these level set is clearly the intersection of M with the plane that passes through the origin. By Golab's theorem, its length is less than eight. We conclude that $\beta(h)$ and $\beta(M)$ are less than eight. ∎

3.4 Corollary. *The unit sphere of a Minkowski space has at least one closed geodesic whose length is less than eight.*

Incredibly, nothing else is known about the behaviour of geodesics on the unit sphere of a Minkowski space.

Problem. *Does there exist a Minkowski space whose unit sphere has a closed geodesic of length less than six?*

4. Covariant derivatives

The two definitions of normals in a Minkowski space yield two different ways of defining a covariant derivative on a surface. Let us start with the covariant derivative coming from the osculation normals.

If $M \subset \mathbf{R}^3$ is an embedded surface and m is a point in M, we denote by $P_m : T_m\mathbf{R}^3 \to T_mM$ the projection along the osculation normal of T_mM.

Let $\gamma : [a, b] \to M$ be a smooth curve and let $V; [a, b] \to TM$ be a vector field along γ. This means that for each value of the parameter t, $V(t) \in T_{\gamma(t)}M$. We may consider V as a vector valued function and define

$$\frac{DV}{dt}(t) = P_{\gamma(t)}\frac{dV}{dt}(t) \ .$$

4.1 Definition. The vector field V along the curve γ is said to be *parallel* if $\frac{DV}{dt}(t) \equiv 0$. The curve γ is said to be a *geodesic* if the vector field $\dot\gamma$ along γ is parallel.

4.2 Definition. Let $\gamma : [a, b] \to M$ be a smooth curve and let v be a vector in $T_{\gamma(a)}M$. The parallel transport of v along γ is the vector $V(b) \in T_{\gamma(b)}M$, where V is the parallel vector field along γ with $V(a) = v$.

This definition of covariant derivative, simple and natural, is useless. Its main problem is that geodesics for the connection do not correspond to geodesics for the metric on M.

Problem. *Let M be a surface and let ∇ be a torsion-free affine connection on M. Give necessary and/or sufficient conditions for ∇ to be the connection induced by an embedding of M into a Minkowski space.*

Let us now give another definition of covariant derivative based on the definition of v-normals at the end of last lecture.

If $v \in T_mM$ is a tangent vector with unit norm, we denote by $\mathbf{P}_v : T_m\mathbf{R}^3 \to T_mM$ the projection along the v-normal of T_mM.

If $\gamma : [a,b] \to M$ is a smooth curve, which we assume to be parametrized by arclength, and $V : [a,b] \to TM$ is a vector field along γ we define

$$\frac{DV}{dt}(t) = \mathbf{P}_{\dot\gamma(t)}\frac{dV}{dt}(t)\ .$$

4.3 Proposition. *A smooth curve $\gamma \subset M$ is a (metric) geodesic parametrized by arclength if and only if $\frac{D\dot\gamma}{dt} \equiv 0$.*

Proof. This follows immediately from proposition 3.1. ∎

The covariant derivative we just defined has a subtle problem : it does not induce an affine connection on TM. However, it does induce an affine connection on another bundle. To explain this let us extend somewhat the definition of our covariant derivative.

Let $\sigma : [a,b] \to TM$ be a smooth curve such that for every t, the vector $\sigma(t)$ has unit length. If $V : [a,b] \to TM$ is a vector field along the curve $\pi \circ \sigma$, we define

$$\frac{DV}{dt}(t) = \mathbf{P}_{\sigma(t)}\frac{dV}{dt}(t)\ .$$

We shall see that this covariant derivative induces an linear connection on the bundle $\pi^*TM \to UM$, where the base consists of unit tangent vectors to M and the fiber over a unit vector $v \in T_mM$ is exactly T_mM.

4.4 Definition. Let $s : UM \to \pi^*TM$ be a section and let X be vector tangent to UM. Let $\sigma : (-\epsilon, \epsilon) \to UM$ be a smooth curve with $\dot\sigma(0) = X(\sigma(0))$, and consider $V := s \circ \sigma$ as a vector field along the curve $\pi \circ \sigma$. We define $\nabla_X s := \frac{DV}{dt}(0)$.

4.3 Exercise. Show that ∇ defines a linear connection on $\pi^*TM \to SM$.

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Lecture 5. Approaches and Examples in Finsler Geometry

In the following two lectures we define Finsler manifolds and cover some of the major approaches to the subject. We also present many examples of Finsler manifolds. Many of these examples are new and exhibit behaviour which is quite different from that of Riemannian manifolds.

1. Basic concepts

1.1 Definition. Let M be a smooth manifold and let $TM \setminus 0$ denote the set of nonzero tangent vectors of M. A *Finsler metric* on M is a smooth positive function $\varphi : TM \setminus 0 \longrightarrow \mathbf{R}$ with the following properties:

- $\varphi(\lambda v_m) = |\lambda| \varphi(v_m)$, for any nonzero real number λ and any nonzero vector $v_m \in T_m M$.
- For each $m \in M$, the set $\{v_m \in T_m M : \varphi(v_m) = 1\}$ is a quadratically convex hypersurface.

A Minkowski space is itself a Finsler manifold where the function φ is invariant under translations. Any surface in a Minkowski space inherits a Finsler structure as does any submanifold of a Finsler manifold. Later in the lecture, when we outline the major approaches to the study of Finsler manifolds, we will treat a great quantity of examples of Finsler metrics with special properties

1.2 Definition. Let (M, φ) be a Finsler manifold. We define the *length* of of a smooth curve $\gamma : [a, b] \to M$ as the integral $\int_a^b \varphi(\dot\gamma(t))\, dt$. The *distance*, $d(x, y)$, between two points x and y on M is defined as the infimum of the lengths of all smooth curves on M joining these two points.

5.1 Exercise. Having defined the distance between two points, we can redefine the length of a curve $\gamma : [a, b] \to M$ by the formula

$$\text{length of } \gamma = \sup\{\sum_{i=0}^{k-1} d(\gamma(t_i), \gamma(t_{i+1})) : a = t_0 < \cdots < t_k = b \text{ is a partition of } [a, b]\} .$$

Show that both definitions of length agree.

A Finsler manifold is above all a metric space. Classical metric invariants such as diameter, or less classical ones such as the *k-extent* :

$$xt_k(M) := \sup\{ \binom{k}{2}^{-1} \sum_{i<j} d(x_i, x_j) : x_1, \ldots, x_k \in M\}$$

are then interesting quantities to study in Finsler geometry.

1.3 Definition. A continuous curve on a metric space M is called a *segment* if its length equals the distance between its endpoints. It is called a *geodesic* if it is locally a segment.

The study of geodesics on a general metric space is hopeless. It could almost be said that Finsler geometry are exactly those metric spaces for which geodesics are well-behaved. The key to this 'wll-behavedness' is that in Finsler geometry geodesics are also solutions of a variational problem :

1.4 Proposition. *A curve $\gamma : [a,b] \to M$ on a Finsler manifold (M,φ) is a geodesic if and only it is an extremal of the functional*

$$l(\sigma) := \int_a^b \varphi(\dot\sigma(t))dt \ ,$$

where σ ranges over all smooth curves $\sigma : [a,b] \to M$ with $\sigma(a) = \gamma(a)$ and $\sigma(b) = \gamma(b)$.

As a result of this proposition, geodesics satisfy the *Euler-Lagrange equations* which in local coordinates $(q,\dot q)$ take the form :

$$\frac{d}{dt}\frac{\partial\varphi}{\partial\dot q} - \frac{\partial\varphi}{\partial q} = 0 \ .$$

5.2 Exercise. A curve $\gamma : [a,b] \to M$ on a Finsler manifold (M,φ) is an extremal of the *action functional*

$$A(\sigma) := \frac{1}{2}\int_a^b \varphi^2(\dot\sigma(t))dt \ ,$$

if and only if γ is a geodesic and $\varphi(\dot\gamma(t))$ is constant.

In what follows we shall always assume that geodesics are parametrized by arclength. In this case it is more convienient to use the action functional.

To solve the following exercises take a book in Riemannian geometry ([Kl] or [Do], for instance) and verify that the proofs and concepts extend to Finsler geometry.

5.3 Exercise. Let M be a Finsler manifold and let m be a point in M. there exists a number $\epsilon > 0$ such that any pair of points in the ball of radius ϵ around m can be joined by a minimizing geodesic contained wholly inside this ball. Moreover, the interval of definition of geodesics starting from a point p inside the ball is at least equal to the distance from p to the boundary of the ball.

5.4 Exercise. Define the exponential map in a Finsler manifold and show that if $m \in M$ and $r > 0$ are such that $\exp_m$ is defined on the set of vectors $\{v_m \in T_mM : \varphi(v_m) < r\}$, then every point p whose distance from m is less than r can be joined to m by a minimizing geodesic.

5.5 Exercise. Prove the Finsler *Hopf-Rinow* theorem: A Finsler manifold is a complete metric space if and only if every geodesic is defined for all time t. Morever, in this case every two points can be joined by a segment.

2. Busemann's G-spaces

In [Bu] Busemann abstracts the properties of geodesics of a Finsler manifold and defines a special class of metric spaces which he calls *G-spaces*.

2.1 Definition. A metric space M is a *G-space* if it satisfies the following properties:

- The space M is locally compact, i.e., every bounded infinite subset has an accumulation point.
- Given two distinct points x and z, there exists a third point y different from them and such that $d(x,y) + d(y,z) = d(x,z)$.
- For every point $p \in M$ there exists a number ϵ such that if x and y are two points with $d(p,x)$ and $d(p,y)$ less than ϵ, then there exists a point z such that $d(x,y) + d(y,z) = d(x,z)$.
- If x, y, z_1 and z_2 are points such that $d(x,z_1) = d(x,y)+d(y,z_1) = d(x,y)+d(y,z_2) = d(x,z_2)$, then $z_1 = z_2$.

5.6 Exercise. Prove that a Finsler manifold is a G-space.

As a consequence of the previous exercise, the study of G-spaces contains the study of Finsler manifolds. Indeed, it contains an important part; Busemann's synthetic approach remains one of the most geometric and powerful approaches to Finsler geometry.

To give a flavor of Busemann's work we shall present some of his definitions and theorems. For proofs we refer the reader to [Busemann].

2.2 Theorem. *Let M be a 2-dimensional G-space. If for every pair of points there is a unique geodesic passing through them, then M is homeomorphic to the projective plane or to a proper subset of it. Moreover, if all the geodesics are closed then M is homemorphic to the projective plane and all geodesics have the same length.*

2.3 Theorem. *If γ is a noncontractible closed curve in a compact G-space M, then M has a closed geodesic homotopic to γ.*

2.4 Definition. A G-space M is said to have *nonpositive curvature* if for every point p there exists a positive number ϵ such that every geodesic triangle $\triangle xyz$ contained in the ball of radius ϵ around p satisfies the following condition : the distance between the midpoints of the segments xy and xz is less than or equal to half the distance between the points y and z.

Busemann also defines G-spaces of *zero curvature* or *negative curvature* by changing the above condition to an equality or a strict inequality.

2.5 Theorem. *If M is a G-space with nonpositive curvature, the distance function $d : M \times M \to$ $\mathbf{R}$ is locally convex.*

5.7 Exercise. Show that a Minkowski space is a G-space with zero curvature.

To construct a compact G-space with zero curvature just take a square in a Minkowski plane and identify the opposite side to obtain a torus. In general any G-space which is locally isometric to a Minkowski space has zero curvature.

2.6 Theorem. *If a Finsler manifold has zero curvature in the sense of Busemann, then it is locally isometric to a Minkowski space.*

3. Variational calculus

We will now consider the concepts and techniques which calculus of variations puts at our disposal for the study of Finsler geometry. We shall also give some examples and pose some problems which are motivated by the variational approach. In the next lecture we will complement the material in this section by considering Finsler metrics from the Hamiltonian viewpoint.

Curvature of Finsler surfaces

Elementary considerations in variational calculus allow us to define an important and computable invariant of a Finsler surface (M, φ). This invariant, the *curvature*, is based on the rate at which nearby geodesics diverge from each other and differs from its Riemannian analogue in that it is defined on the unit bundle of M, UM, and not on M.

3.1 Definition. Given a geodesic $\gamma : [a, b] \to M$, a *variation of γ through geodesics* is a smooth map $\Gamma : (-\epsilon, \epsilon) \times [a, b] \to M$ such that

(1) $\Gamma(0, t) = \gamma(t)$.

(2) For each fixed s_0, the curve $t \mapsto \Gamma(s_0, t)$ is a geodesic.

3.2 Definition. Let $\Gamma(s,t)$ be a variation of γ through geodesics. The vector field Y along γ defined by

$$Y(t) = \frac{\partial}{\partial s}|_{s=0}\Gamma(s,t)$$

is called a *Jacobi field.* A Jacobi field is said to be *nontrivial* if, for each t, $Y(t)$ lies on the acceleration normal to the line spanned by $\dot{\gamma}(t)$.

We are interested only on nontrivial Jacobi fields, since we do not want to consider trivial variations such as $\Gamma(s,t) = \gamma(t+s)$.

3.3 Definition. Let p be a point in a Finsler surface and let γ be a geodesic starting from p. A point $\gamma(s)$ is said to be *conjugate* to p along γ if there exists a nonzero Jacobi field $Y(t)$ along γ such that $Y(0) = Y(s) = 0$.

3.4 Proposition. *Let p be a point in a Finsler surface and let γ be a geodesic starting from p. The curve γ restricted to the interval $[0,s]$ minimizes length if and only if there is no conjugate point between $p = \gamma(0)$ and $\gamma(s)$.*

If we are given a geodesic γ let us orient the tangent spaces $T_{\gamma(t)}M$ and let us defined $J\dot{\gamma}(t)\}$ in such a way that $\{\dot{\gamma}(t), J\dot{\gamma}(t)\}$ is an oriented orthonormal basis for the Euclidean structure on $T_{\gamma(t)}M$ defined by the osculating ellipse to the unit circle at the point $\dot{\gamma}(t)$. In other words, the vector $J\dot{\gamma}(t)\}$ lies on the acceleration normal o the line spanned by $\dot{\gamma}(t)$ and has unit length for the quadratic form $D^2L(\dot{\gamma}(t))$.

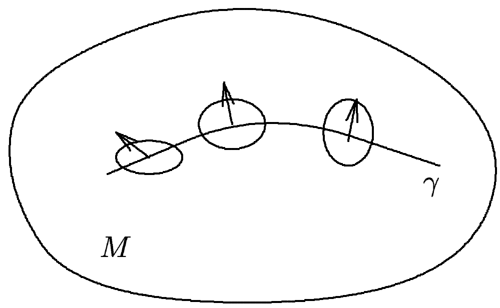


Fig. 5.1. The vector field $J\dot{\gamma}(t)$ along γ.

Any Jacobi field Y along γ can be uniquely expressed as

$$Y(t) = y(t)J\dot{\gamma}(t),$$

where y is a real-valued function.

The infinitesimal srpeading of geodeiscs is measured the convexity of the function $|Y(t)| = |y(t)|$, which in turn is measured by the sign of the function

$$K(t) = -\frac{y''(t)}{y(t)} .$$

3.5 Theorem-Definition. *There is a unique smooth function $\mathbf{K} : UM \to \mathbf{R}$ such that for any unit speed geodesic γ, any proper Jacobi field $Y(t) = y(t)J\dot{\gamma}(t)$ satisfies the Jacobi equation*

$$y''(t) + \mathbf{K}(\dot{\gamma}(t))y(t) = 0.$$

The function $\mathbf{K}$ is called the *curvature* of (M, φ). In contrast with the Riemannian case, $\mathbf{K}$ depends on both the point $\gamma(t) \in M$ and the direction of $\dot{\gamma}(t)$.

The proof of the theorem will be postponed until lecture 7, where we will study in detail the structure of the unit tangent bundle of a Finsler structure.

Here are two results which easily follow from the Jacobi equation :

3.6 Theorem (Pedersen). *The curvature function of a Finsler surface (M, φ) is nonnegative if and only if the distance function $d : M \times M \to \mathbf{R}$ is locally convex.*

3.7 Theorem. *If the curvature function of a Finsler surface (M, φ) is greater than or equal to a positive number δ, then the diameter of M is less than or equal to $\pi/\sqrt{\delta}$.*

Existence and number of closed geodesics

The first question that comes to mind from the point of view of the calculus of variations is that of the existence and behavior of closed geodesics. In the Riemannian case this question has a long and glorious history with contributors like Poincare, Birkhoff, and Morse. However, there do not seem to be (at least for the moment) really interesting results about closed geodesics in Finsler manifolds. The reason is that if an existence result in Riemannian geometry depends only on Morse theory and the method of broken geodesics, then it holds in Finsler geometry. We have then basically the same theorems and no new insight is gained. The following two theorems illustrate our point :

3.8 Theorem. *Any compact Finsler manifold has at least one closed geodesic.*

3.9 Theorem. *Any Finsler metric on the two-dimensional sphere has at least three simple closed geodesics.*

Fortunately, not all existence theorems on periodic geodesics are Morse-theoretic. For example, a recent theorem of Franks states that any Riemannian metric on the two-dimensional sphere has infinitely many closed geodesics. The proof cannot be readily adapted to general Finsler metrics and we are left with a nice problem :

Problem. *Does every Finsler metric on the two-dimensional sphere have infinitely many closed geodesics ?*

Inverse problems

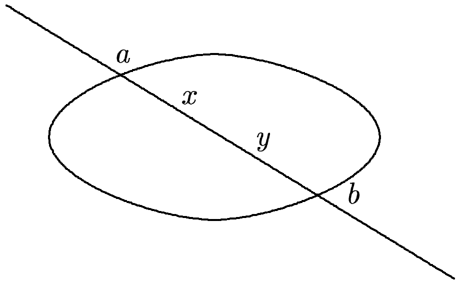
Another source of problems and theorems in Finsler geometry is that of constructing and characterizing Finsler metrics prescribed geodesics. The most famous of these problems is, without doubt, Hilbert's fourth problem :

Hilbert's fourth problem. *Characterize and study all Finsler metrics on $\mathbf{RP}^n$ and its convex subsets for which geodesics are projective lines.*

At the root of this problem is the following beautiful geometric construction also due to Hilbert.

Example : Hilbert geometries.

Let C be a closed convex curve on the plane and let D denote its interior. If x and y are two distict points on D we let a and b be the points of intersection of C with the line determined by x and y (see figure 5.2).



$$d(x,y) := \ln\left(\frac{||y-a||}{||x-a||}\frac{||x-b||}{||y-b||}\right) .$$

Fig. 5.2. Hilbert's geometries.

3.10 Theorem (Hilbert). *The function d is a distance function on D. Moreover, straight line segments are geodesics.*

Proof. The only nontrivial part in the proof that d is a distance function is verifying the triangle inequality for triplets of noncollinear points. Let us take z is to be a point on D and which is not in the line passing through x and y. Since the logarithm is an increasing function all we have to show is that

$$\frac{||z-c||}{||x-c||}\frac{||x-d||}{||z-d||} < \frac{||y-a||}{||x-a||}\frac{||x-b||}{||y-b||}\cdot\frac{||y-f||}{||z-f||}\frac{||z-e||}{||y-e||} .$$

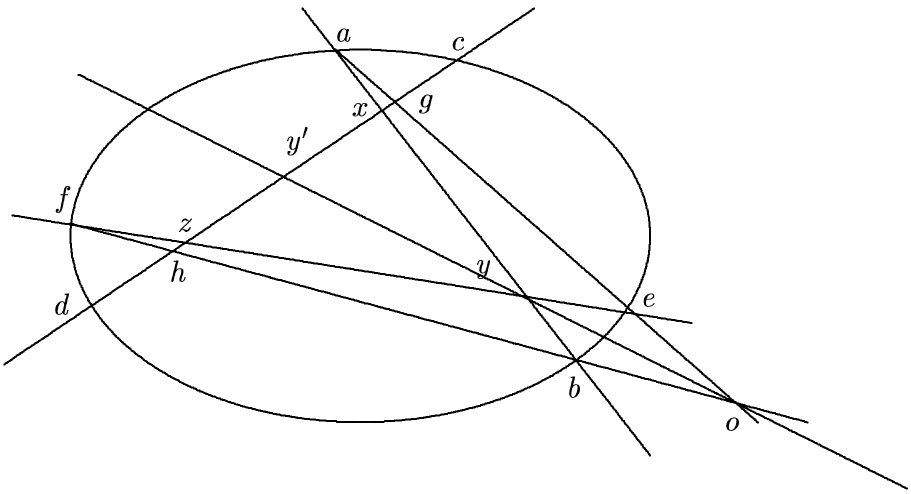


Fig. 5.3. Proof of the triangle inequality.

Note that this quantities are cross-ratios and, therefore, projectively invariant. To make use of this fact draw the line ae, the line bf, and the point of intersection o. Project the points a, b, e, f, and y to the line xz as in figure 5.2 to obtain the points g, h, and y'. The invariance of the cross-ratio under projective transformations implies that

$$\frac{||y-a||}{||x-a||}\frac{||x-b||}{||y-b||}\cdot\frac{||y-f||}{||z-f||}\frac{||z-e||}{||y-e||} = \frac{||y'-g||}{||x-g||}\frac{||x-h||}{||y'-h||}\cdot\frac{||y'-h||}{||z-h||}\frac{||z-g||}{||y'-g||}$$
$$= \frac{||x-h||}{||x-g||}\frac{||z-g||}{||z-h||} .$$

We leave the verification that $\frac{\|z-c\|}{\|x-c\|}\frac{\|x-d\|}{\|z-d\|} < \frac{\|x-h\|}{\|x-g\|}\frac{\|z-g\|}{\|z-h\|}$ to the industrious reader.

To show that are straight lines segments in D are geodesics, it is sufficient to note that if z is a point in the interior of the segment xy, then $d(x,y) = d(x,z) + d(y,z)$. Using this fact and the definition of length given in the first part of the lecture, it is obvious that the length of the segment xy equals the distance between x and y. ∎

The relation between Hilbert geometries and Finsler manifolds is given by the following beautiful description which we learned from R. Ambartzumian.

5.8 Exercise. Show that if C is a closed curve which is quadratically convex, then Hilbert's metric is actually a Finsler metric. Hint : If $x \in D$ and $v \in T_x D$, then let $\varphi(v) := t_1^{-1} + t_2^{-1}$, where $x + t_1 v$ and $x - t_2 v$ belong to C.

An interesting property of Hilbert geometries when seen as Finsler manifolds is the following result of Funk :

3.11 Proposition. *The Hilbert geometries are Finsler surfaces with constant curvature equal to* -1.

Hamel, a student of Hilbert, was the first to seriously study Hilbert's fourth problem. Among many things he showed that Lagrangians on $\mathbf{R}^n$ for which extremals are straight lines are characterized by a simple system of linear partial differential equations.

3.12 Theorem (Hamel). *Let* $\varphi : T\mathbf{R}^n \setminus 0 \to \mathbf{R}$ *be a smooth function which is homogeneous of order one. Straight lines are extremals of the functional* $l(\gamma) := \int \varphi(\dot\gamma(t))dt$ *if and only if* φ *satisfies the following system of equations :*

$$\frac{\partial^2\varphi}{\partial q_i \partial \dot q_j} = \frac{\partial^2\varphi}{\partial q_j \partial \dot q_i} \ , \ \text{for } 1 \le i,j \le n \ .$$

Proof. Recall that the Euler-Lagrange equations for the functional l are

$$\frac{d}{dt}\frac{\partial\varphi}{\partial \dot q_i} - \frac{\partial\varphi}{\partial q_i} = 0 \ \text{,for } 1 \le i \le n \ .$$

Since $\frac{\partial\varphi}{\partial q_i}$ is homogeneous of degree one in the velocities, we have that

$$\frac{\partial\varphi}{\partial q_i}(x,v) = \sum_{j=1}^{n} \frac{\partial^2\varphi}{\partial q_i \partial \dot q_j}(x,v)\cdot v_j$$

and hence we can write the Euler-Lagrange equations as

$$\frac{d}{dt}\frac{\partial\varphi}{\partial \dot q_i} - \sum_{j=1}^{n} \frac{\partial^2\varphi}{\partial q_i \partial \dot q_j}\cdot \dot q_j = 0 \ , \ \text{for } 1 \le i \le n \ .$$

Note that when we take a line $t \mapsto (x + tv, v)$,

$$\frac{d}{dt}\frac{\partial\varphi}{\partial \dot q_i}(x+tv, v) = \sum_{j=1}^{n} \frac{\partial^2\varphi}{\partial q_j \partial \dot q_i}\cdot v_j$$

and that the Euler-Lagrange equations become

$$\left(\sum_{j=1}^{n} \frac{\partial^2 \varphi}{\partial q_j \partial \dot{q}_i} - \sum_{j=1}^{n} \frac{\partial^2 \varphi}{\partial q_i \partial \dot{q}_j} \right) \cdot v_j = 0 \ .$$

This equation holds for every straight line $t \mapsto (x + tv, v)$ if and only if

$$\frac{\partial^2 \varphi}{\partial q_i \partial \dot{q}_j} = \frac{\partial^2 \varphi}{\partial q_j \partial \dot{q}_i} \ , \text{ for } 1 \leq i, j \leq n \ . \quad \blacksquare$$

5.9 Exercise. Show that if φ_1 and φ_2 are two Finsler metrics on $\mathbf{R}^n$ such that their geodesics are projective lines, then any convex combination $(1-t)\varphi_1 + t\varphi_2$, $0 \leq t \leq 1$, is a Finsler metric with the same property.

Functions $\varphi : T\mathbf{R}^n \setminus 0 \to \mathbf{R}$ which are homogeneous of order one and which satisfy Hamel's equations have a beautiful integral representation which was (probably) discovered by Pogorelov.

3.13 Theorem. *A function $\varphi : T\mathbf{R}^n \setminus 0 \to \mathbf{R}$ which is homogeneous of order one satisfies Hamel's equations if and only if there exists a smooth even function $\nu(r, \xi)$ on $\mathbf{R} \times S^{n-1}$ such that*

$$\varphi(q, \dot{q}) = \int_{\xi \in S^{n-1}} |\xi \cdot \dot{q}| \nu(\xi \cdot q, \xi) \Omega \ ,$$

where Ω is the standard area form on the unit sphere in $\mathbf{R}^n$.

5.10 Exercise. Show that if the function ν is positive, then φ is a Finsler metric. Careful! the converse is only true in the plane (see [Alvarez-Gelfand-Smirnov]).

Thanks to the theorem and the exercise, we see that it is extremely easy to construct Finsler metrics on $\mathbf{R}^n$ whose geodesics are straight lines.

5.11 Exercise. Show that the Finsler metric on the plane obtained from the function $\nu(r, \theta) := 1 + r^2$ on the cylinder $\mathbf{R} \times S^1$ is

$$\varphi(x_1, x_2, v_1, v_2) := \frac{4}{3\sqrt{v_1^2 + v_2^2}} \cdot (2x_1 x_2 v_1 v_2 + (3 + 2x_1^2 + x_2^2) v_1^2 + (3 + 2x_2^2 + x_1^2) v_2^2) \ .$$

The preceeding example is taken from [Alvarez-Fernandes] and, to be honest, we did use *Mathematica*.

To construct Finsler metrics on $\mathbf{RP}^n$ whose geodesics are projective lines we make use of a smooth volume density on the dual projective space $\mathbf{RP}^{n*}$. If $xy \subset \mathbf{RP}^n$ is a line segment with endpoints x and y, we define its length to be the volume of all hyperplanes $\Pi \in \mathbf{RP}^n$ which intersect it. We define the distance between x and y to be the minimum of the length of the two line segments which join them.

Exercise 5.10. Show that the preceeding construction defines a metric on $\mathbf{RP}^n$ and that projective lines are geodesics.

If the reader thinks that Hilbert's fourth problem was too easy, the authors propose the following

Problem. *Construct all Finsler metrics on* $\mathbf{CP}^n$ *such that the geodesics coincide with the geodesics of the standard Riemannian metric on* $\mathbf{CP}^n$.

In trying to solve the preceeding problem, the authors came up with a construction of Finsler metrics on $\mathbf{CP}^n$ such that the geodesics are circles. In particular, we construct metrics on the two-dimensional sphere such that its geodesics are circles, but not necessarily great circles. Here is the construction:

Example. Finsler metrics on $\mathbf{CP}^n$ whose geodesics are circles.

To understand the construction it is necessary to remember that in the Hopf fibration $\rho : S^{2n+1} \to \mathbf{CP}^n$ the projections of all great circles in S^{2n+1} are circles in $\mathbf{CP}^n$.

5.12 Exercise. Prove it!

We construct a Finsler metric on S^{2n+1} all of whose geodesics are great circles by first constructing a Finsler metric ϕ on $\mathbf{RP}^{2n+1}$ whose geodesics are projective lines and then lifting it to the sphere by the canonical covering map from S^{2n+1} to $\mathbf{RP}^{2n+1}$. We may construct this metric in such a way that the circle action of the Hopf fibration acts by isometries : just choose a smooth volume density in $\mathbf{RP}^{n*}$ which is invariant with respect to the induced action.

We now define the Finsler metric φ on $\mathbf{CP}^n$ by defining the hypersurfaces $S_z := \{v_z \in T_z\mathbf{CP}^n : \varphi(v_z) = 1\}$ as follows :

Take a point $x \in S^{2n+1}$ with $\rho(x) = z$ and let S_z be the boundary of the image of the hypersurface $\{w_x \in T_x S^{2n+1} : \phi(w_x) = 1\}$ under the linear projection $D\rho : T_x S^{2n+1} \to T_z\mathbf{CP}^n$. The symmetry of the metric ϕ implies that S_z does not depend on the choice of the point x with $\rho(x) = z$.

3.14 Theorem ([AD]). *The geodesics of the Finsler metric* φ *are circles.*

Remark. A theorem of Cartan (see [Cartan]) implies that any Riemannian metric on $\mathbf{CP}^n$, $n > 1$, such that the $\mathbf{CP}^1$'s are totally geodesic is a multiple of the standard Riemannian metric on $\mathbf{CP}^n$. The behaviour of the geodesics in the examples above is then a typically Finsler phenomenon.

For a proof of this theorem and for a fuller description of the above construction we refer the reader to [Alvarez-Durán].

We end the section with two problems we do not yet know how to solve.

Problem. *Is there any Riemannian metric on* S^2 *which is not isometric to a multiple of the standard metric and such that all of its geodesics are circles?*

Problem. *Construct all Finsler metrics on* $\mathbf{CP}^n$ *whose geodesics are circles.*

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Lecture 6. Approaches and Examples in Finsler Geometry (cont.)

1. Hamiltonian mechanics

In the last lecture we considered Finsler metrics as variational problems which we then treated using the *Lagrangian formalism*. In the present lecture we will change the point of view to the *Hamiltonian formalism* which is more geometric and in many instances more powerful.

Basic concepts.

As we remarked in the first part of the previous lecture, the geodesics of a Finsler manifold are extremals of the action functional

$$A(\gamma) := \frac{1}{2} \int \varphi^2(\dot{\gamma}(t))dt \ .$$

The function $L := \frac{1}{2}\varphi^2$ is called the *Lagrangian*. Fixing the base point m we have that for each nonzero $v \in T_mM$ the quadratic from $D^2L(v)$ as defined in Lecture 1 is a positive definite quadratic form on T_mM.

Let $m \in M$ be a point and consider the restriction of L to the fiber T_mM. If v_m is a tangent vector, then dL is a vector in T^*_mM. Repeating this construction at each fiber we obtain a smooth map $FL : TM \setminus 0 \to T^*M \setminus 0$. This is just the paprametrized version of the Legendre transform defined in Lecture 1 and it is also known as the *Legendre transform.*

6.1 Exercise. Use the nondegeneracy of the bilinear forms $D^2L(v), v \in TM \setminus 0$ to show that FL is a diffeomorphism.

The inverse of the map FL and the Lagrangian L can be used to define the *Hamiltonian function* $H : T^*M \setminus 0 \to \mathbf{R}$ by the equation $H(p_m) := L(FL^{-1}(p_m))$.

A simpler way to define the Hamiltonian of a Finsler metric is given in the following exercise :

6.2 Exercise. Show that H is the unique function on $T^*M \setminus 0$ which is homogeneous of order two on the fibers and such that for each point $m \in M$, the set $\{p_m \in T^*_mM : H(p_m) = 1\}$ is the dual convex hypersurface to $\{v_m \in T_mM : \varphi(v_m) = 1\}$.

The advantage of considering the Hamiltonian instead of the Lagrangian is that we are able to use the natural symplectic structure on the cotangent bundle to study the dynamic properties of the Finsler metric.

1.1 Definition. Let M be a smooth manifold and let $\pi : T^*M \to M$ be its cotangent bundle. The *canonical* 1*-form* α on T^*M is defined by the equality $\alpha(v_{p_m}) = p_m(D\pi(v_{p_m}))$. The 2-form $\omega := -d\alpha$ is called the *canonical* 2*-form* on T^*M.

6.3 Exercise. Show that the canonical 2-form is nondegenerate in the sense that if $v_{p_m} \in T_{p_m}T^*M$ is such that $\omega(v_{p_m}, w_{p_m}) = 0$ for all vectors $w_{p_m} \in T_{p_m}T^*M$, then $v_{p_m} = 0$.

A differential 2-form which is closed and nondegenerate is called a *symplectic form*. The previous exercise shows that the cotangent bundle of any manifold has a natural symplectic form.

If M is an n-dimensional manifold, the form $\Omega = \omega^n$ is a volume form which we call the *symplectic volume form* on T^*M. Using this volume form we can reap our first benefit from the Hamiltonian formalism : a natural definition of the volume of a Finsler manifold.

1.2 Definition. The *volume* of an n-dimensional Finsler manifold (M, L) is defined to be the symplectic volume of the set

$$D^*M := \{p_m \in T^*M : H(p_m) < 1\}$$

divided by the volume of the euclidean n-dimensional ball. The *k-area* of a k-dimensional submanifold of M equals its volume with the induced Finsler metric.

6.4 Exercise. Show that this definition coincides with the standard definition in the case of Riemannian manifolds and with the Holmes-Thompson definition in the case of submanifolds of Minkowski spaces.

In Riemannian geometry there are deep inequalities involving the volume of a multiply connected surface and the length of its shortest noncontractible geodesic. These inequalities are called *systolic inequalities* (for more information see chapter 4 in [Gromov]). In the Finsler case the validity of these, or similar, inequalities seems to be open.

Problem. *Find the infimum, taken over all Finsler metrics φ on $\mathbf{RP}^2$, of the volume of $(\mathbf{RP}^2, \varphi)$ divided by the square of the length of the shortest noncontractible geodesic.*

Problem. *Find the infimum, taken over all Finsler metrics φ on the torus T^2, of the volume of (T^2, φ) divided by the square of the length of the shortest noncontractible geodesic.*

Problem. *Let M be an orientable surface of genus g and let $C(g)$ be the infimum, taken over all Finsler metrics φ on the M, of the volume of (M, φ) divided by the square of length of the shortest noncontractible geodesic. Is $C(g)$ a positive number?*

If the infima in the preceeding exercises are taken over the set of all Riemannian metrics in the respective surfaces, then a theorem of Pu states the answer to the first problem is $2/\pi$ with equality if and only if the Riemannian metric on $\mathbf{RP}^2$ is the standard, a theorem of Lowner states that the answer to the second problem is $\sqrt{3}/2$ with equality if and only if is the metric is that of the flat equilateral torus, and a theorem of Acolla and Blatter answers the last problem affirmatively.

Now we turn back to the basics of Hamiltonian mechanics.

1.3 Definition. If $H : T^*M \to \mathbf{R}$ is a smooth function,we define its *Hamiltonian vector field* X_H to be the vector field on T^*M such that for any other vector field V on T^*M $dH(V0 = \omega(X_H, V)$.

1.4 Theorem. *If H is the Hamiltonian of a Finsler metric, then the integral curves of X_H project onto geodesics of the Finsler metric.*

Idea of the proof. It is simple to show that if $\gamma(t)$ is a geodesic which, as usual in these notes, is parametrized by arclength, then the curve $FL(\dot\gamma(t))$ is an integral curve of X_H. Since FL leaves the base points fixed, this immediately implies the theorem. ∎

6.5 Exercise. Use the previous theorem and the Hopf-Rinow theorem in Finsler geometry to show that if the Finsler manifold (M, φ) is a complete metric space, then the vector field X_H defines a flow on $T^*M \setminus 0$.

1.5 Theorem. *The function H is constant along the integral curves of X_H.*

Proof. All we have to do is to show that $dH(X_H) = 0$ and this follows immediately from the equation $\omega(X_H, X_H) = dH(X_H)$ and the skew-symmetry of ω. ∎

In what follows we shall assume our Finsler manifolds to be complete and, by exercise 6.5, that the vector field X_H defines a flow $\Phi : \mathbf{R} \times T^*M \setminus 0 \to T^*M \setminus 0$. This flow is called the *geodesic flow* of the Finsler metric.

1.6 Theorem. *The geodesic flow of any Finsler metric preserves the symplectic form.*

Proof. By Cartan's formula for the Lie derivative of a form along a vector field we have that

$$L_{X_H}\omega = d(\omega\rfloor X_H) + d\omega\rfloor X_H \ .$$

To see that this is zero just note that $(\omega\rfloor X_H) = dH$ and that $d\omega = 0$. ∎

An important consequence of the preceeding result is that the geodesic flow preserves the symplectic volume form.

Using the transformation $FL : TM \setminus 0 \to T^*M \setminus 0$ we can pullback the symplectic form ω to a symplectic form ω_L on $TM \setminus 0$. Using ω_L and the Lagrangian L we can define the vector field X_1 on $TM \setminus 0$ by the equality $\omega_L(X_1, \cdot) = dL(\cdot)$. The vector field X_1 will be one of our stars in the next lecture

6.6 Exercise. Show that the flow of X_1 preserves both the function L and the form ω_L.

6.7 Exercise. Show that if γ is a geodesic, then $\dot\gamma$ is an integral curve of the vector field X_1.

We are now close to the middle of the lecture with nothing to show for it except a natural definition of volume, a few interesting problems, and some exercises. It's time make things up by giving a nice construction of Finsler metrics on the sphere and by proving that the volume of the unit sphere of a Minkowski space equals that of its dual.

Symplectic equivalence of Finsler metrics.

1.7 Definition. Two Finsler manifolds M_1 and M_2 with Hamiltonians H_1 and H_2 are said to be *s-equivalent* if D^*M_1 and D^*M_2 are symplectomorphic.

Clearly, s-equivalent Finsler manifolds have the same volume. The following examples and results on s-equivalence are taken from [Al] and [AF]. We also refer the reader to the last paper for all the proofs to the assertions that follow.

Example. Let S be a centrally symmetric convex hypersurface in $\mathbf{R}^n$ and let $S^{n-1} \subset \mathbf{R}^n$ be the Euclidean unit sphere. For each point q on the sphere consider T_qS^{n-1} as an affine hyperplane of $\mathbf{R}^n$. Define a Finsler metric φ_S on S^{n-1} by requiring that the unit tangent sphere $\{v_q \in T_qS^{n-1} : \varphi_S(v_q) = 1\}$ be the boundary of the image of S under orthogonal projection onto T_qS^{n-1}.

1.8 Theorem. *The Finsler manifold (S^{n-1}, φ_S) is s-equivalent to the convex hypersurface S with the Riemannian metric inherited from its embedding into $\mathbf{R}^n$.*

This theorem was first proved by A. Reznikov (unpublished) and then rediscovered by the first author in [Al]. The result suggests the following problem :

Problem. *Is every Finsler manifold s-equivalent to a Riemannian manifold?*

1.9 Theorem ([AF]). *Let $(V, \|\cdot\|)$ be a Minkowski space and let $(V^*, \|\cdot\|^*)$ be its dual. The unit spheres of V and V^* with their inherited Finsler metrics are s-equivalent.*

As a corollary we get the following result of Holmes and Thompson which was mentioned in lecture 4.

Corollary ([HT]). *The volume of the unit sphere of a Minkowski space and that of its dual are equal. In particular, the length of the unit circle of a Minkowski plane equals the length of the unit circle of its dual.*

We now show our hand by stating the simple and general result from which follow the previous theorems. For the rest of the course we shall use the notation $[W]$ to denote the translation to the origin of an affine subspace W.

Theorem [AF]. *Let V be a vector space and let $S \subset V$ and $\Sigma \subset V^*$ be smooth strictly convex hypersurfaces. The interiors of the sets*

$$\{(v, \xi_{|[T_v S]}) : v \in S, \xi \in \Sigma\} \subset T^*S \text{ and } \{(\xi, v_{|[T_\xi \Sigma]}) : \xi \in \Sigma, v \in S\} \subset T^*\Sigma$$

are symplectomorphic.

When S is a centrally symmetric convex hypersurface and Σ is an euclidean sphere also centered at the origin we have Reznikov's theorem, when Σ is the dual of S we have that the unit sphere of a Minkowski space and its dual are s-equivalent. In general, if $S_1 := S$ and $S_2^* := \Sigma$ are centrally symmetric convex hypersurfaces we have an interesting duality principle.

Theorem [AF]. *Let S_1 and S_2 be two centrally symmetric convex surfaces in a vector space V. The Finsler metric on S_1 induced from the embedding of S_1 into the Minkowski space (V, S_2) is s-equivalent to the Finsler metric on S_2^* induced from the embedding of S_2^* into the Minkowski space (V^*, S_1^*).*

Corollary [HT]. *Let S_1 and S_2 be two centrally symmetric convex surfaces in a vector space V. The Finsler metric on S_1 induced from the embedding of S_1 into the Minkowski space (V, S_2) and the Finsler metric on S_2^* induced from the embedding of S_2^* into the Minkowski space (V^*, S_1^*) have the same volume.*

The s-equivalence of Finsler metrics gives much more than just the equality of the volumes. Deep results in symplectic geometry by Hofer and Floer imply that the set of lengths of closed geodesics of s-equivalent Finsler metrics is the same. It would be interesting to know if there are other metric invariants which are preserved.

Problem. *Is the diameter of two s-equivalent compact Finsler manifolds necessarily the same ?*

4.3. Integrability of geodesic flows. We now turn to the study of the dynamics of the geodesic flow. Because we have nothing interesting to say in higher dimensions, we restrict ourselves to the case of Finsler surfaces.

1.10 Definition. Let M be a surface and let H be a smooth function on T^*M. The flow of the vector field X_H is said to be *integrable* if there exists a smooth function F on T^*M such that F is constant along the integral curves of X_H and $dH \wedge dF$ is different from zero almost everywhere. The function F is said to be an *integral of motion*.

A very convenient way to express that a function F is an integral of motion is through the *Poisson bracket*:

Exercise 6.8. Show that a function F is an integral of motion for the flow of X_H if and only if the *Poisson braket* $\{H, F\} := \omega(X_H, X_F)$ is identically zero.

The Arnold-Liouville theorem is the cornerstone of the theory integrable Hamiltonian system. Here we just particularize the Arnold part of this theorem to the case at hand.

1.11 Theorem. *Let M be a surface and let H and F be two smooth functions on T^*M one of which is proper and such that their Poisson bracket is identically zero. If (x, y) is a regular value of the map $(H, F) : T^*M \to \mathbf{R}^2$, then the intersection of the level sets $H^{-1}(x)$ and $F^{-1}(y)$ is a union of disjoint tori and the flow of either X_H or X_F restricted to any one of these tori is conjugate to a linear flow.*

Note that Hamiltonians of a Finsler surface are proper functions since they are positive and homogeneous of order two in the fibers of the cotangent bundle. Indeed, the homogeneity condition together with the fact the the vector field X_H is tangent to the set $H = 1$ implies that the study the geodesic flow is the same as the study its restriction to $H = 1$. The following exercise precises why it is possible to restrict our attention to $H = 1$ when studying the integrability of the geodesic flow of a Finsler metric.

Exercise 6.9. Let F be a smooth function defined on the submanifold $H = 1$ which is constant on the integral curves of X_H and such that dF is nonzero almost everywhere. Show that if we extend F to $T^*M \setminus 0$ as a homogeneous function of order two, then $\{H, F\} = 0$ and $dH \wedge dF$ is nonzero almost everywhere.

Integrability of the geodesic flow imposes restrictions on the topology of the Finsler surface. The following result is a consequence of Fomenko's topological classification of constant energy surfaces of integrable Hamiltonian systems (see [Fo]).

1.12 Theorem. *Let M be a compact orientable Finsler surface with Hamiltonian H. If there exists a Morse-Bott function on $H = 1$ which is constant along the integral curves of X_H, then M is diffeomorphic to either the torus or the sphere.*

Example. Some integrable Finsler metrics on the sphere. The construction we shall now present is purely geometrical, but we think it will not overly tax the readers patience.

Let S^2 be the euclidean unit sphere on $\mathbf{R}^3$ and let $S \subset \mathbf{R}^3$ be a smooth centrally symmetric convex surface. If q is a point in S^2 we denote by $[T_qS^2]$ the plane passing through the origin and parallel to the plane tangent to S^2 at q.

Let us denote the intersection of $[T_qS^2]$ with the convex surface S by $C(q)$ and let us take the parametrization $\gamma(t)$ of $C(q)$ such that the vector product $\gamma(t) \times \dot{\gamma}(t)$ is constantly equal to q. The curve $\dot{\gamma}(t)$ defines a convex curve on $[T_qS^2]$ which we denote by $\hat{C}(q)$

We define a Finsler metric φ on S^2 by requiring that, upon translation to the origin, $\{v_q \in T_qS^2 : \varphi(v_q) = 1\}$ be the convex curve $\hat{C}(q)$.

1.13 Theorem ([AD]). *For any smooth centrally symmetric convex surface S the above construction yields an integrable Finsler metric on the sphere. Moreover if H_0 is the Hamiltonian of the standard Riemannian metric on S^2, then H_0 is an integral of motion.*

Sketch of proof for experts. Consider the action of the rotation group $SO(3)$ on S^2. Lift this action to a Hamiltonian action on T^*S^2 and consider the momentum map $\mu : T^*S^2 \to so(3)^*$. Let $f : so(3)^* \setminus 0 \to \mathbf{R}$ be a smooth positive function which is homogeneous of degree 2 and such that $f^{-1}(1)$ is quadratically convex. The function $f \circ \mu$ on $T^*S^2 \setminus 0$ is the Hamiltonian of a Finsler metric on the sphere. By construction it Poisson commutes with H_0 which equals the composition of the Casimir on $so(3)^*$ with the momentum map.

By using the standard identification of $so(3)^*$ with $\mathbf{R}^3$, the identification of T^*S^2 and TS^2 given by the standard metric, and the Legendre transform, we obtain the geometric construction given above. ∎

For a fuller description of this integrable Finsler metrics the reader is refered to [AD].

2. Convex geometry

A Finsler manifold can also be thought as a family of centrally symmetric convex hypersurfaces parametrized by the points of the manifold. Many interesting problems arise relating the geometric invariants of the hypersurfaces and the topology of the manifold. Let us begin justifying ourselves by defining some invariants of Finsler surfaces.

In the first lecture we defined a centro-affine invariant of centrally symmetric convex curves which we called **I**. We now define an invariant of the same name as the function on the unitary bundle of a Finsler surface (M, φ) whose value, at a unit vector $v_m \in T_m M$, equals the value of **I** of the convex curve $\{w_m \in T_m M : \varphi(w_m) = 1\}$ at v_m. Clearly, **I** is identically zero if and only if the Finsler surface is actually Riemannian.

One interesting function of the invariant **I** proposed by Shen is as an obstruction to the isometric embedding of a Finsler surface in a Minkowski space. Indeed, if a Finsler surface can be isometrically embedded in a Minkowski space, then **I** must be bounded.

Problem. *Can any complete Finsler surface whose invariant* **I** *is bounded be isometrically embedded in some Minkowski space of sufficiently large dimension?*

In the first lecture we also defined the total angle of a centrally symmetric convex curve S. The function which assigns to each point m of M the total angle of the unit tangent circle $\{w_m \in T_m M : \varphi(w_m) = 1\}$ is also an invariant of M.

Problem. *Construct a Finsler metric on the two-dimensional sphere for which the total angle of its unit tangent circles is a constant.*

This problem is of great importance since only for metrics where the total angle of the unit circles is constant is there a Gauss-Bonnet theorem (see [Chern-Bao]). Up to now the only compact surfaces on which such metrics have been constructed are the torus and the Klein bottle.

Example. Ginsler manifolds.

2.1 Definition. A Finsler manifold (M, φ) is said to be Ginsler if for any two points x and y in M, there is a linear map from $T_x M$ to $T_y M$ taking the unit tangent sphere at x to the unit tangent sphere at y.

Ginsler surfaces are examples of Finsler surfaces for which the total angle of its unit tangent circles is a constant. It is easy to construct many examples of Ginsler metrics on the torus or the Klein bottle by using the fact that their tangent bundles are trivializable : just choose a locally Minowski metric φ on the surface and a linear transformation $A_m : T_m M \to T_m M$ changing smoothly in m. The metric $\varphi_A(v_m) := \varphi(A_m \cdot v_m)$ is Ginsler.

Unfortunately there are no more examples in two dimensions:

2.2 Proposition. *A Ginsler metric on a compact surface with nonzero Euler characteristic is Riemannian.*

In general, if an n-dimensional manifold admits a non Riemannian Ginsler metric the structure group of its tangent bundle can be reduced to a subgroup of $SO(n)$ which does not act transitively on the spherized tangent bundle. In the case of surfaces this condition implies that the tangent bundle is trivial.

2.3 Proposition. *If a compact Ginsler surface cannot be isometrically embedded in a Minkowski 3-space, then the space is Euclidean and the surface Riemannian.*

Proof. If we isometrically embed a compact Ginsler surface into a three-dimensional Minkowski space $(V, \|\cdot\|)$, then all intersections of the unit sphere with planes passing through the origin are linearly equivalent. By a result in lecture 4 this implies that $(V, \|\cdot\|)$ is a Eulidean space. ∎

A third invariant, which will play a major role in the next lecture, is not a function but a vector field on UM which we denote by X_3. To define X_3 we give an *angular parametrization* of each unit circle $U_m M$ and let X_3 be the velocity field. The vector field X_3 is tangent to the fibers of the bundle $UM \to M$ and its flow defines a circle action if and only if the total angle of the unit circles is constant.

All the invariants we have defined so far in this section make use of only the convex geometry of the unit circles at each tangent space without taking into account how this geometry changes from one point to another. More interesting would be to have invariants which measure this change. One such invariant has a truly simple definition :

Let $v_m \in T_m M$ be a unit tangent vector and let $\gamma : (-\epsilon, \epsilon) \to M$ be the geodesic with initial condition v_m. Define $\mathbf{J}(v_m) = d/dt\, \mathbf{I}(\dot\gamma(t))_{|t=0}$.

The invariant $\mathbf{J}$ is then a smooth function on the unit bundle of M which measures how $\mathbf{I}$ changes along the geodesics. Note that $\mathbf{J} = 0$ means that $\mathbf{I}$ is an invariant of motion.

2.4 Definition. A Finsler surface for which the invariant $\mathbf{J}$ is identically zero is called a *Lansberg surface* .

Unfortunately all the known examples of Landsberg surfaces are rather trivial. For example, the following problem remains open :

Problem. *Is there a Landsberg metric on the sphere ?*

The following result is an interesting remark made to one of the authors by G. Paternain.

2.5 Proposition. *Let (M, φ) be a Landsberg surface which is compact and orientable. If the invariant $\mathbf{I}$ is a Morse-Bott function, then M is diffeomorphic to either the torus or the sphere.*

Proof. This follows immediately from the fact that $\mathbf{I}$ is an integral of motion and from the theorem of Fomenko mentioned in the last section. ∎

Another interesting property of Landsberg surfaces which will be proved in the next lecture is the following result due to Landsberg :

2.6 Proposition. *If (M, φ) is a Landsberg surface, then the total angle of its unit tangent circles is a constant.*

This will be proved in the next lecture.

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Lecture 7. Cartan's Structure Equations

1. Relations between the basic invariants

In lectures 5 and 6 we defined three important invariants of Finsler surfaces : $\mathbf{I}, \mathbf{J}$ and $\mathbf{K}$. We recall that $\mathbf{I}$ is a convex geometric invariant which describes the shape of each unit tangent circle, $\mathbf{J}$ measures how $\mathbf{I}$ changes along geodesics, and $\mathbf{K}$ is a variational invariant which measures the focusing of geodesics. All three invariants are functions on the unit bundle of the Finsler surface.

The purpose of this lecture is to show, following Cartan, how these invariants are related. For the most part we follow the pqper [Br] where, besides a clear exposition on Cartan's approach to Finsler geometry, the reader will many interesting examples of Finsler surfaces.

Recall that in lecture 6 we defined two tangent vector fields in the unit bundle and which we called X_1 and X_3. The vector field X_1 is completely determined by the geodesic flow : if v_m is a unit tangent vector and $\gamma(t)$ is a geodesic with initial condition v_m, then $X_1(v_m) := d/dt\ \dot{\gamma}(t)_{|t=0}$. The vector field X_3 measures the angle in the unit tangent circles.

1.1 Theorem ([Ca]). *If we define $X_2 := [X_3, X_1]$, then we have the following equations :*

$$\begin{aligned}
[X_3, X_1] &= X_2, \\
[X_1, X_2] &= \mathbf{K}X_3, \\
[X_3, X_2] &= -X_1 + \mathbf{I}X_2 + \mathbf{J}X_3 \ .
\end{aligned}$$

Of course, Cartan prefered differential forms to vector fields and he wrote the above equations in terms of the forms ω_1, ω_2 and ω_3 defined by the equations $\omega_i(X_j) = \delta_{ij}$. Cartan's *structure equations* are :

$$\begin{aligned}
d\omega_1 &= -\omega_2 \wedge \omega_3, \\
d\omega_2 &= \omega_1 \wedge \omega_3 - \mathbf{I}\omega_2 \wedge \omega_3, \\
d\omega_3 &= -\mathbf{K}\omega_1 \wedge \omega_2 - \mathbf{J}\omega_2 \wedge \omega_3 \ .
\end{aligned}$$

In [Ca] approach Cartan first finds the forms ω_1, ω_2 and ω_3 and shows that they verify the above equations for *some* functions $\mathbf{I}$, $\mathbf{J}$, and $\mathbf{K}$. He then explains the geometric significance of the invariants. We have preferred to begin with a more detailed description of the basic invariants of Finsler surfaces and to state Cartan's them as a nearly miraculous relation between them. Note that by differentiating the structure equations we obtain the following *Bianchi identities* :

$$\begin{aligned}
\mathbf{J} &= \mathbf{I}_1, \\
\mathbf{K}_3 + \mathbf{K}\mathbf{I} + \mathbf{J}_1 &= 0 \ .
\end{aligned}$$

In these identities the subindices represent differentiation with respect to the vector fields X_1, X_2 and X_3. In general if F is a function on the unit bundle we will write

$$dF = F_1\omega_1 + F_2\omega_2 + F_3\omega_3 \ .$$

The derivation of Cartan's structure equations is presented in at least three references (see [Ca], [Ch], and [Br]) so rederiving them here would only bring us bad luck. What we prefer to do is to share some of our insights and to give some interesting applications of the structure equations.

The definition of X_2 as the Lie bracket of X_3 and X_1 is not specially enlightening, but it can be motivated as follows :

If $\pi : UM \to M$ is the natural projection of the unit bundle onto M and $v_m \in UM$ is any point, then $D\pi(X_1(v_m)) = v_m$ and $D\pi(X_3(v_m)) = 0$. It would be nice to have have a vector field X on UM such that at every point $v_m \in UM$, v_m and $D\pi(X(v_m)$ form an orthonormal basis of T_mM for the Euclidean metric given by the osculating ellipse to the unit tangent circle at v_m.

Note that once we have constructed the vector field X we have many others like it simply by adding multiples of X_3. A way to choose from this infinite number of vector fields is to require that $[X_1, X]$ be a multiple of X_3. This determines X up to a sign.

1.2 Proposition. *The vector field $X_2 := [X_3, X_1]$ is such that for any point $v_m \in UM$, v_m and $D\pi(X_2(v_m))$ an orthonormal basis of T_mM for the Euclidean metric given by the osculating ellipse to the unit tangent circle at v_m. Moreover, X_2 is, up to a sign, the only vector field with this property and such that $[X_1, X_2]$ is a multiple of X_3.*

7.1 Exercise. Prove proposition 1.2.

It is not trivial to see that the invariants coming out of the structure equations are exactly those defined in the previous lectures. The authors know that the invariant $\mathbf{I}$ agrees for the unenlightening reason that the formula for $\mathbf{I}$ given in exercise 1.8 agrees with that given by Cartan in [Ca] once the appropriate changes of notations are taken into account. Indeed, all we did was to extract the geometry from Cartan's formula. The fact that $\mathbf{J}$ also agrees follows from the Bianchi identities. A bit more interesting is to prove that $\mathbf{K}$ agrees as well :

1.3 Proposition. *The invariant $\mathbf{K}$ from the structure equations equations satisfies the definition of the curvature function : for any proper variation $\Gamma(s,t)$ with variational vector field $Y(t) = y(t)J\dot\gamma(t)$, the function $y(t)$ satisfies the Jacobi equation $y''(t) + \mathbf{K}(\dot\gamma(t))y(t) = 0$.*

Proof. Let $\gamma : [a,b] \to M$ be a geodesic and let $\Gamma : (-\epsilon,\epsilon) \times [a,b] \to M$ be a proper variation of γ through geodesics, with variational vector field $Y(t) = y(t)J\dot\gamma(t)$ as in lecture 5. We define the *velocity lift* of Γ as the function $\Lambda : (-\epsilon,\epsilon) \times [a,b] \to UM$ given by

$$\Lambda(s,t) = \frac{\partial\Gamma}{\partial t}(s,t).$$

7.2 Exercise. Prove that the velocity lift satisfies the following properties :

(1) $D\pi\partial\Lambda/\partial s = \partial\Gamma/\partial s$.

(2) $\partial\Lambda/\partial t = X_{1_{\Lambda(s,t)}}$.

(3) For a proper variation Λ parametrizes a smooth surface in UM, even at the points in which $Y(t) = 0$; these points just get "unrolled" along the fibers.

(4) $y(t) = \omega_2(\partial\Lambda/\partial s)|_{s=0}$.

By (4), we have that $y''(t) = \frac{\partial^2}{\partial t^2}\omega_2(\partial\Lambda/\partial s)|_{s=0}$. Let's show that this equals $-\mathbf{K}(\dot\gamma(t))y(t)$.

Claim. $y'(t) = -\omega_3(\partial\Lambda/\partial s)_{|t=0}$

Indeed, on suppressing the evaluation at $t = 0$, we have that

$$\begin{aligned} y'(t) = \frac{\partial}{\partial t}\omega_2(\partial\Lambda/\partial s) =& d\omega_2(\partial\Lambda/\partial s, \partial\Lambda/\partial t) + \\ & \frac{\partial}{\partial s}\omega_2(\partial\Lambda/\partial t) + \omega_2([\partial\Lambda/\partial s, \partial\Lambda/\partial t]) \\ =& d\omega_2(\partial\Lambda/\partial s, \partial\Lambda/\partial t) \\ =& -\omega_3(\partial\Lambda/\partial s) \ , \end{aligned}$$

where the last equality comes from the structure equations.

Similarly, $y''(t)$ is given by

$$\begin{aligned} \frac{\partial^2}{\partial t^2}\omega_2(\partial\Lambda/\partial s) &= -\frac{\partial}{\partial t}\omega_3(\partial\Lambda/\partial s) \\ &= d\omega_3(\partial\Lambda/\partial s, \partial\Lambda/\partial t) + \frac{\partial}{\partial s}\omega_3(\partial\Lambda/\partial t) \\ &+ \omega_3([\partial\Lambda/\partial s, \partial\Lambda\ /\partial t]) \\ &= d\omega_3(\partial\Lambda/\partial s, \partial\Lambda/\partial t) \\ &= -\mathbf{K}\omega_2(\partial\Lambda/\partial s) \ . \end{aligned}$$

We have then that the coefficient $\mathbf{K}$ in the structure equations satisfies the definition of the curvature function. ∎

2. Applications of Cartan's structure equations

Let us begin our tour of applications by an interesting result of Akbar-Zadeh

2.1 Theorem ([Ak]). *If (M, φ) is a compact Finsler surface of constant negative curvature, then M is Riemannian.*

Proof. The second Bianchi identity tells us that $\mathbf{K}_3 + \mathbf{K}\mathbf{I} + \mathbf{J}_1 = 0$. If $\mathbf{K} \equiv c$ is a constant and $\gamma(t)$ is a geodesic on M, the function $\mathbf{I}(t) := \mathbf{I}(\dot\gamma(t))$ satisfies the differential equation

$$\frac{d^2}{dt^2}\mathbf{I} = -c\mathbf{I} \ .$$

If c is negative, then $\mathbf{I}(t)$ must be a linear combination of exponentials. If the initial condition is not $\mathbf{I}(0) = 0, \mathbf{I}'(0) = 0$, then $\mathbf{I}(t)$ is unbounded.

Since $\mathbf{I}$ is bounded whenever the Finsler surface is compact, the only possibility that remains is that $\mathbf{I}$ be identically zero, and that th surface be Riemannian. ∎

The fact that the invariant $\mathbf{I}$ is unbounded for non Riemannian Finsler surfaces of constant negative curvature has other interesting implications.

2.2 Theorem. *The Hilbert geometry given by a convex curve C does not admit an isometric embedding into a Minkowski space unless C is an ellipse.*

Proof. If C is not an ellipse, then we have a non Riemannian Finsler surface with curvature constantly equal to -1 and therefore the invariant $\mathbf{I}$ is unbounded. As remarked during the last lecture, the invariant $\mathbf{I}$ of a Finsler surface which is isometrically embedded in a Minkowski space is bounded. It follows that the Hilbert geometry given by C cannot be embedded in any Minkowski space. ∎

Note that if $\mathbf{K} \equiv 0$, then along any geodesic $\gamma(t)$, $\mathbf{I}'' = 0$ and so t is linear as a function of t. If the surface is compact, then necessarily $\mathbf{I}$ is a constant along every geodesic. The problem of the global classification of compact surfaces with zero curvature is treated at length in [Br].

2.3 Proposition ([Br]). *If (M, φ) is a Finsler surface with constant curvature equal to one, then the function $\mathbf{I}^2 + \mathbf{J}^2$ is constant along geodesics.*

Proof. From the Bianchi identities it follows that $\mathbf{J}_1 = -\mathbf{I}$ and we know that $\mathbf{I}_1 = \mathbf{J}$. We have then that $X_1(\mathbf{I}^2 + \mathbf{J}^2) = 2\mathbf{I}\mathbf{I}_1 + \mathbf{J}\mathbf{J}_1 = 0$. ∎

All the previous applications involve only the Bianchi equations and not the structure equations *per se*. Here is one application that does. The proof is taken from [Br].

2.4 Proposition. *Let (M, φ) be a Landsberg surface. If M is connected, then the total angle of any two of its tangent unit circles is the same.*

Proof. If x and y be two points on M, the difference of the total angle of the unit circle S_y over y and the total angle of the unit circle S_x over x is given by

$$\int_{S_y} \omega_3 - int_{S_x} \omega_3 \ ,$$

where the orientation over S_y and S_x is taken so that the integrals are positive.

Let $\gamma : [0, 1] \to M$ be a smooth curve joining them. Let $\pi : UM \to M$ denote the natural projection and set $\mathcal{C}$ be the cylinder $\pi^{-1}(\gamma)$. Note that the oriented boundary of $\mathcal{C}$ is $S_y - S_x$ and that the 2-form $\omega_1 \wedge \omega_2$ vanishes identically on $\mathcal{C}$.

Using Stokes theorem and the structure equations we have that

$$\int_{S_y} \omega_3 - int_{S_x} \omega_3 = \int_{\mathcal{C}} d\omega_3 = \int_{\mathcal{C}} -\mathbf{K}\omega_1 \wedge \omega_2 - \mathbf{J}\omega_2 \wedge \omega_3 = 0 \ . \quad ∎$$

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