SOME PROBLEMS ON FINSLER GEOMETRY

J.C. ÁLVAREZ PAIVA

Abstract. This article contains the statements and motivations of thirty-one problems on Finsler geometry and related fields. It aims to show that recent results in convex geometry, the calculus of variations, symplectic geometry, and integral geometry can be powerful tools in the study of Finsler manifolds and may be used to overcome the legacy of Riemann's poor opinion of the field.

We do like intuitive geometric arguments and uncovering simple geometric reasons underlying seemingly recondite facts.
H. Busemann.

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1. Introduction

Finsler manifolds, manifolds whose tangent spaces carry a norm which varies smoothly with the base point, were born prematurely in 1854 together with their Riemannian counterparts in Riemann's ground-breaking *Habilitationsvortrag*. I say prematurely because in 1854 Minkowski's work on normed spaces and convex bodies (see [54]) was still forty three years away, and thus not even the infinitesimal geometry on which Finsler manifolds are based was understood or appreciated at the time. Apparently, Riemann did not know what to make of these 'more general class' of manifolds whose element of arclength does not originate from a scalar product and, fatefully, put in a bad word for them:

> Investigation of this more general class would actually require no essential different principles, but it would be rather time-consuming and

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> throw relatively little new light on the study of Space, especially since the results cannot be expressed geometrically.

Given the awe in which we rightfully regard Riemann's achievements and uncanny geometrical intuition, it is tempting to take the above quotation out of historical context and to dismiss Finsler geometry altogether. But, if we think of the great advances in convex geometry, the calculus of variations, integral geometry, the theory of metric spaces, and symplectic geometry that have taken place since 1854, then we may be moved to reassess Riemann's statement and to consider applying these new tools to develop the subject in a way that Riemann could not have forseen.

To aid the reader in this reassessment, the present paper includes thirty-one simply-stated open problems, as well as a survey of the more elementary and geometric chapters of Finsler geometry. It presents a detailed discussion of the notion of volume and area in Finsler manifolds with a strong bias towards the so-called Holmes-Thompson definition which, because of its symplectic nature, is easier to work with than the Hausdorff measure. The other highlights of the paper are its presentation of Hilbert's fourth problem and its elementary approach to the differential invariants of Finsler surfaces. These are mostly based on the papers [14, 12, 13] with I.M. Gelfand, M. Smirnov, and E. Fernandes, as well as on the lecture notes [10] written jointly with C. Durán.

In view of the often-made criticisms of Finsler geometry — very few concrete and interesting examples, very few non-Riemannian theorems of real geometric content, and too many subindices — I have tried to include as many concrete examples, simply-stated results, and geometric constructions as possible. In this way, many of the jewels, so to speak, of Finsler geometry find their way into the following pages.

As anyone writing a survey paper, I have had to make some choices. In matters of taste, I have consistently prefered the concrete to the abstract, the elementary to the advanced, the C^∞ to the C^k, and the global to the local. I have stayed clear of Riemann-Finsler geometry and Finsler connections because the book [20] of Bao, Chern, and Shen covers the subject in depth as as do the lecture notes of Abate and Patrizio ([1]). Because of my ignorance of the subject, I have not touched on complex Finsler geometry (see [1] also for this topic) and, since I do not want to give up the metric aspects of Finsler manifolds, I have chosen not to treat nonsymmetric Finsler metrics. A tough choice was not to include anything on Busemann's G-spaces. This approach, which consists in abstracting the properties of geodesics on Finsler manifolds, is one of the most powerful in Finsler geometry, but it is impossible to outdo Busemann's own exposition in [34, 35].

The reader should keep in mind that this is a biased survey of Finsler geometry and, although it provides adequate background and references to study some of the problems it contains, it does not give a complete picture of the activity in the field. Highly recommended are the surveys of Chern ([39]) and Busemann ([32]) whose opposing points of view give much food for thought. The proceedings [19] contains many open problems and a vista of various approaches to Finsler geometry. The book [65] contains a beautiful exposition of the convex-geometric aspects of the Holmes-Thompson volume as well as most of the convex geometry necessary for the study of Finsler manifolds. The lecture notes [10] are similar in spirit to the present paper. The thesis of Egloff ([41]) is a good place to learn about the beautiful results of Egloff and Foulon on the geometry and dynamics of Finsler manifolds with non-positive and negative curvature. Finally, I wholeheartedly recommend looking at the papers [37] and [3] before plunging into other papers where Finsler connections are treated.

The reader can find many of the preprints cited in this paper, along with other works on the interaction between convex, integral, metric, and symplectic geometry, in the site of the Finsler geometry newsletter: `http://gauss.math.ucl.-ac.be/~fweb`.

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2. Preliminaries

If $(V, \|\cdot\|)$ is a real, finite-dimensional normed space, we define the length of a smooth curve $\gamma : [a, b] \to V$ by the formula

$$\text{length of } \gamma := \int_a^b \|\dot{\gamma}(t)\| dt.$$

A smooth submanifold $N \subset V$ inherits a metric from the norm: if x and y are two points on N, define their distance as the infimum of the lengths of all smooth curves on N joining x and y. Notice that in order to define the metric on N it suffices to know the restriction of the norm to each tangent space. This motivates the following heuristic definition: A Finsler manifold is a manifold together with the choice of a norm on each tangent space. The precise definition requires us to restrict the class of norms to those where the unit sphere is smooth and has positive principal curvatures for some (and therefore any) auxiliary Euclidean structure. The intrinsic definition of these norms is as follows:

Let V be a vector space and let $\varphi : V \to [0, \infty)$ be a norm which is smooth outside the origin. If v is a nonzero vector, we define the bilinear form $g_\varphi(v)$ evaluated at a pair of vectors w_1 and w_2 by taking a smooth vector valued function $\alpha(s,t)$ such that $\alpha(0,0) = v$, $\frac{\partial \alpha}{\partial s}(0,0) = w_1$, $\frac{\partial \alpha}{\partial t}(0,0) = w_2$, and setting

$$g_\varphi(v)(w_1, w_2) := \frac{1}{2}\frac{\partial^2 \varphi^2}{\partial s \partial t}(0,0).$$

Definition 2.1. A smooth norm $\varphi : V \to [0, \infty)$ is said to be a *Minkowski norm* if for every nonzero vector v, the bilinear form $g_\varphi(v)$ is positive definite. A finite-dimensional vector space provided with a Minkowski norm will be called a *Minkowski space*.

When the vector v belongs to the unit sphere, we will denote $g_\varphi(v)$ as the *osculating Euclidean structure* at v and the ellipsoid

$$E_v := \{w \in V : g_\varphi(v)(w, w) = 1\}$$

as the *osculating ellipsoid* at v.

Definition 2.2. Let M be a smooth manifold and let $TM \setminus 0$ denote its tangent bundle with the zero section deleted. A *Finsler metric* on M is a smooth function

$$\varphi : TM \setminus 0 \longrightarrow \mathbb{R}$$

such that for each point $m \in M$ the restriction of φ to $T_m M$ is a Minkowski norm.

Some examples of Finsler manifolds are submanifolds of Minkowski spaces and flat tori obtained as quotients of Minkowski spaces.

If $\gamma : [a,b] \to M$ is a smooth curve on a Finsler manifold (M,φ), then the quantity

$$\text{length of } \gamma := \int_a^b \varphi(\dot\gamma(t))dt, \tag{1}$$

is independent of the parametrization. Using this definition of length we define a metric on M by letting the distance between two points $x, y \in M$ to be the infimum of the lengths of all smooth curves joining x and y. Finsler manifolds are *length spaces*: the length of a curve γ defined by the integral in (1) equals the metric length of the curve given by

$$\sup\{\sum_{i=0}^{k-1} \mathrm{dist}(\gamma(t_i), \gamma(t_{i+1})) : a = t_0 < \cdots < t_k = b \text{ is a partition of } [a,b]\}.$$

The condition that the norms in each tangent space be Minkowski norms is necessary for the study of the geodesics. Namely, we want these to be solutions of a second order differential equation on M.

The Hamiltonian point of view

If (V,φ) is a normed space, then the dual vector space V^* inherits a natural norm defined by the equation

$$\varphi^*(\xi) := \sup\{|\xi \cdot v| : \varphi(v) \le 1\}.$$

A related construction on Minkowski spaces is the *Legendre transform* which assigns to a nonzero vector $v \in V$ the covector defined by $g_\varphi(v)(v,\cdot)$. It is easy to check that if v belongs to the unit sphere $S \subset V$, then the Legendre transform of v is the unique covector ξ such that $\xi \cdot w = 1$ for all w belonging to the hyperplane tangent to S at the point v. This implies that the image of S under the Legedre transform is the unit sphere in (V^*, φ^*).

Let (M,φ) be a Finsler manifold and for each point $m \in M$ let φ_m denote the Minkowski norm on T_mM. If (T_m^*M, φ_m^*) is the dual of the normed space (T_mM, φ_m), then the function

$$H : T^*M \longrightarrow \mathbb{R}$$

defined by $H(p_m) := \varphi_m^*(p_m)$ is a Hamiltonian whose energy surfaces are fiberwise convex. Applying the Legendre transform on each tangent space of M allows us to identify $TM \setminus 0$ and $T^*M \setminus 0$, and to pass from φ to H.

By passing from a Finsler metric to its associated Hamiltonian, we gain access to the techniques of Hamiltonian mechanics and symplectic geometry. Below, we recall some of the basic definitions and constructions. For more information, we recommend the reader the texts of Arnold, Abraham, and Marsden ([15] and [2]).

Definition 2.3. Let $\pi : T^*M \to M$ be the standard projection and let $D\pi : T(T^*M) \to TM$ be its differential. The *canonical 1-form* α on T^*M is defined by the equation $\alpha_{p_m}(v_{p_m}) = p_m(D\pi(v_{p_m}))$, where $p_m \in T_m^*M$ and $v_{p_m} \in T_{p_m}(T^*M)$. The *symplectic 2-form* is defined as $\omega := -d\alpha$.

The form ω is nondegenerate: at each point $p_m \in T^*M$, the map $v_{p_m} \mapsto \omega_{p_m}(v_{p_m}, \cdot)$ is an isomorphism from $T_{p_m}(T^*M)$ to $T_{p_m}^*(T^*M)$. We can use this isomorphism to pass from 1-forms on T^*M to vector fields on T^*M.

Definition 2.4. Let $H : T^*M \to \mathbb{R}$ be a smooth function. The *Hamiltonian vector field* of H, X_H, is defined by the equality $dH = \omega(X_H, \cdot)$

As an easy consequence of the definition, we have the following classical result:

Theorem 2.1. *The function H is constant along the integral curves of the Hamiltonian vector field X_H. Moreover, the symplectic form is invariant under the flow of X_H.*

Because of this result, it is usual to disregard the function H in favor of the *unit co-sphere bundle* $S^*_H M := H^{-1}(1)$. If α is the canonical 1-form on T^*M, then its restriction to the unit co-sphere bundle $S^*_H M$, which we denote by α_H, is a *contact form* (i.e., the top-order form $\alpha_H \wedge (d\alpha_H)^{n-1}$ never vanishes). Using α_H, we can define the restriction of the Hamiltonian vector field X_H without any reference to the function H:

Definition 2.5. The *Reeb vector field* X_H on $S^*_H M$ is defined by the equations

$$\alpha_H(X_H) = 1, \quad d\alpha_H(X_H, \cdot) = 0.$$

The projection to M of the integral curves of this vector field are geodesics parametrized with unit speed. Conversely, if γ is a geodesic on M parametrized with unit speed, then the Legendre transform maps the velocity curve $\dot{\gamma}$ to an orbit of the Reeb vector field. We remark that if γ is any smooth curve on M parametrized with unit speed and $\bar{\gamma}$ is the image of $\dot{\gamma}$ under the Legendre transform, then

$$\text{length of } \gamma = \int_{\bar{\gamma}} \alpha_H. \tag{2}$$

Note that the Finsler manifold (M, φ) is geodesically complete (or metrically complete, since it is easy to verify that the Hopf-Rinow theorem extends to the Finsler setting) if and only if the Reeb vector field defines a flow.

Let us finish this section by remarking that the nondegeneracy of the symplectic form ω on T^*M is equivalent to the fact that ω^n, $n = \dim(M)$, is a volume form. This remark will provide us with a natural way to define the volume of a Finsler manifold.

The Riemannian point of view

Finsler manifolds can also be studied from the point of view of Riemannian manifolds and bundles. Indeed, to every unit vector $v_m \in T_m M$ we may associate the inner product $g_\varphi(v)$. In this way, we can define a Riemannian structure on the pullback of the tangent bundle of M to the unit tangent bundle of M. This construction underlies many of the definitions of connections associated to Finsler manifolds.

A variation on this theme is to take a nowhere zero vector field X defined on an open subset $\mathcal{O} \subset M$ and to associate to it the Riemannian metric on $\mathcal{O}$ defined by $m \mapsto g_\varphi(X(m))$. This construction has been used by Shen (see [62]) to give a simple description of the Finsler curvature.

Isometries and isometric embeddings

The definitions of *isometry* and *isometric embedding* between Finsler manifolds (M, φ_M) and (N, φ_N) is the same as for Riemannian manifolds. Namely, an isometry (resp. isometric embedding) is a diffeomorphism (resp. embedding) $f : M \to N$

such that $f^*\varphi_N = \varphi_M$. Unlike Riemannian manifolds, two Finsler manifolds can fail to be isometric because of a single tangent space. For example, if at a point $m \in M$ the *indicatrix*

$$S_m M := \{v_m \in T_m M : \varphi_M(v_m) = 1\}$$

is an ellipsoid while none of the indicatrices of N are ellipsoids, then M and N are not isometric. This remark points at the important role played by the centro-affine geometry of convex hypersurfaces in Finsler geometry.

In [30] Burago and Ivanov showed that any compact Finsler manifold admits an isometric embedding into a finite-dimensional normed space. It is likely that the norm can be chosen to be a Minkowski norm. They also give examples of noncompact Finsler manifolds that cannot be isometrically embedded in any finite-dimensional normed space. Other examples have been given by Shen (see [62]) and by Álvarez and Durán (see proposition 8.4).

If one dares to separate oneself from the strong Riemannian tradition, there are still interesting problems in the theory of isometric embeddings of Finsler manifolds.

Problem 1. It is known that every two-dimensional normed space is isometric to a subspace of $L_1([0,1])$. Does every two-dimensional Finsler manifold admit an isometric embedding into this Banach space?

Notice that any finite-dimensional Euclidean space is isometric to a subspace of $L_1([0,1])$ and so any finite-dimensional Riemannian manifold can be isometrically embedded in this space.

Problem 2 (Durán)**.** Find an explicit way to isometrically embed flat Finsler tori of dimension two into (low-dimensional) Minkowski spaces.

Problem 3 (Durán)**.** Let $(V_1, \|\cdot\|_1)$ and $(V_2, \|\cdot\|_2)$ be two Minkowski spaces, and let $K_1 \subset V_1$ and $K_2 \subset V_2$ be two smooth convex hypersurfaces. Assuming that $f : K_1 \to K_2$ is an isometry, does it follow that f extends to an isometry $T : V_1 \to V_2$?

As is often the case with basic questions in Finsler geometry, the problem above quickly leads to an interesting problem in convex geometry:

Problem 4. Let B_1 and B_2 be two centrally-symmetric convex bodies in $\mathbb{R}^{n+1}$ centered at the origin and let $\mathbb{R}P^{n*}$ denote the set of all hyperplanes in $\mathbb{R}^{n+1}$ passing through the origin. Suppose there exist a diffeomorphism (or homeomorphism) $f : \mathbb{R}P^{n*} \to \mathbb{R}P^{n*}$ such that if ζ is a hyperplane through the origin, then the sections $\zeta \cap B_1$ and $f(\zeta) \cap B_2$ are linearly equivalent. Does it follow that B_1 and B_2 are linearly equivalent?

3. Volume and area in Finsler spaces

The theory of volume and area in normed spaces has long been a major driving force in convex geometry. We mention the Blaschke-Santalo inequality, the Mahler conjecture, the Busemann-Petty problem, the Shephard problem, and the numerous works of Busemann, Ewald, and Shephard on the notions of convexity on Grassmann manifolds as works that either originated from or have applications to the study of volumes and areas in normed spaces.

Defining a volume on a finite-dimensional normed space seems easy enough: a natural volume should be invariant under translations and, by Haar's theorem, it must be a multiple of the Lebesgue measure. However, the choice of this multiple is crucial. To understand this, suppose we have already decided on how to assign those constants to two-dimensional normed spaces. We can now define the area of a two-dimensional polyhedral surface embedded in a 3-dimensional normed space as the area of each face with its induced norm. Making a different choice of constants leads to a completely different way of measuring the area of polyhedral surfaces.

The guiding principle — and the difficulty — in defining volumes on normed spaces is that the choice of a volume for every k-dimensional normed space leads to the definition of the k-area integrand in all higher-dimensional normed spaces and we want the associated variational problem to be as nice as possible.

In the literature one can find many reasonable choices of volume on normed spaces: the Busemann definition, the Holmes-Thompson definition, the Benson definition, and Gromov's mass*. Of these, the Busemann and the Holmes-Thompson definition are the ones that have been studied most (see the references in [65]).

Busemann defined the volume on a finite-dimensional normed space in such a way that the volume of its unit ball equals the volume the Euclidean unit ball of the same dimension. Two important theorems — both consequences of the Brunn-Minkowski inequality — justify his definition.

Theorem 3.1 (Busemann). *The k-area of a submanifold of an n-dimensional normed space, $n > k$, equals its Hausdorff k-measure.*

Theorem 3.2 (Busemann). *Hyperplanes are area-minimizing on any finite-dimensional normed space. Moreover, the $(n-1)$-area integrand in an n-dimensional Minkowski space is elliptic in the sense of geometric measure theory.*

An outstanding open problem is whether flats of higher codimension are also area-minimizing for the Hausdorff measure. In combinatorial terms, the problem is most appealing:

Problem 5 (Busemann). Let P be a compact polyhedron of dimension k in a normed space. Is the (Busemann) area of any given face less than or equal to the sum of the areas of the remaining faces?

In [50], Holmes and Thompson defined the volume on a finite-dimensional normed space in such a way that the volume of its unit ball equals its volume product divided by the volume of the Euclidean unit ball of the same dimension. We recall the definition of the volume product:

Definition 3.1. Let K be a convex body centered at the origin of an n-dimensional vector space V and let $K^* \subset V^*$ be its dual (or polar) body. Let ω be the symplectic form on $V \times V^* = T^*V$ and let ω^n be the symplectic volume form. The *volume product* of K equals the symplectic volume of $K \times K^* \subset V \times V^*$.

Like in the case of the Busemann volume, the justification for the Holmes-Thompson definition depends greatly on how far can we go in answering the following question:

Problem 6 (Thompson). Let P be a compact polyhedron of dimension k in a normed space. Is the (Holmes-Thompson) area of any given face less than or equal to the sum of the areas of the remaining faces?

In this respect, Holmes and Thompson (see [50] and [65]) proved the following result:

Theorem 3.3 (Holmes and Thompson). *Hyperplanes are area-minimizing on any finite-dimensional normed space. Moreover, the $(n-1)$-area integrand in an n-dimensional Minkowski space is elliptic in the sense of geometric measure theory.*

To describe the next breakthrough, we need to describe *zonoids* and *hypermetric spaces*.

Definition 3.2. A *zonotope* is a polytope all of whose faces are centrally symmetric. A *zonoid* is a Hausdorff limit of zonotopes. A normed space is said to be *hypermetric* if the unit ball of its dual is a zonoid.

These short definitions do not do justice to the richness of the concepts. More information and references can be found in the survey article of Goodey and Weil ([45]).

Theorem 3.4. *Let P be a compact polyhedron of dimension k in a hypermetric normed space. The (Holmes-Thompson) area of any given face is less than or equal to the sum of the areas of the remaining faces*

As will be shown below, this theorem is an easy consequence of the results of Schneider and Wieacker on the integral geometry of hypermetric normed spaces.

The latest progress on the question of minimality of flats in finite-dimensional normed spaces is the annoucement in [31] of the following theorem:

Theorem 3.5 (Burago and Ivanov). *Let P be a compact 2-dimensional polyhedron in a normed space. If P is homeomorphic to a sphere, then the (Holmes-Thompson) area of any given face is less than or equal to the sum of the areas of the remaining faces*

The Hausdorff measure and the Holmes-Thompson volume are related by the classical Blaschke-Santalo inequality which we rewrite in the present context:

Theorem 3.6. *The Holmes-Thompson volume of the unit ball of a finite-dimensional normed space is no greater than the volume of the Euclidean unit ball of the same dimension. Equality holds if and only if the norm is Euclidean.*

This immediately implies that the Hausdorff measure of an open subset of a normed space is greater than or equal to its Holmes-Thompson volume.

One of the main justifications for adopting the Holmes-Thompson volume comes from its role in integral geometry (see [61, 6, 12]). The state of the art is represented by the following two results:

Theorem 3.7 (Schneider and Wieacker, [61]). *Let $(V, \|\cdot\|)$ be an n-dimensional hypermetric space and let k, $1 \leq k \leq n-1$, be an integer. There exists a translation-invariant measure Φ_{n-k} on the manifold $H_{n,n-k}$ of $(n-k)$-flats of V such that if $N \subset V$ is an immersed k-dimensional submanifold, then*

$$vol_k(N) = \frac{1}{\epsilon_k} \int_{\lambda \in H_{n,n-k}} \#(N \cap \lambda) \Phi_{n-k}, \tag{3}$$

where ϵ_k is the volume of the Euclidean unit ball of dimension k.

Theorem 3.8 (Álvarez and Fernandes, [12]). *Let $(V, \|\cdot\|)$ be an n-dimensional Minkowski space and let k, $1 \leq k \leq n-1$, be an integer. There exists a smooth, translation-invariant, and possibly signed measure Φ_{n-k} on the manifold $H_{n,n-k}$ of*

$(n-k)$-flats of V such that if $N \subset V$ is an immersed k-dimensional submanifold, then

$$vol_k(N) = \frac{1}{\epsilon_k} \int_{\lambda \in H_{n,n-k}} \#(N \cap \lambda) \Phi_{n-k}, \tag{4}$$

where ϵ_k is the volume of the Euclidean unit ball of dimension k.

Armed with theorem 3.7, we can easily prove that flats are area-minimizing in finite-dimensional hypermetric spaces. Indeed, if D is a k-flat domain in an n-dimensional hypermetric space and D' is a competitor with the same boundary, then any $(n-k)$-flat intersecting D will intersect D'. It follows from formula 3 and the non-negativity of the measure Φ_{n-k} that the k-area of D is less than or equal to the k-area of D'.

In [13] Álvarez and Fernandes proved that formula 4 is equivalent to a remarkably simple expression for the Holmes-Thompson k-area integrand in terms of the Fourier transform of the norm:

Fourier transforms of norms. Let ϕ be a smooth, even homogeneous function of degree one on an n-dimensional vector space V, let $e_1, \dots, e_n$ be a basis of V, and let $\xi_1, \dots, \xi_n$ be the dual basis in V^*. The basis $e_1, \dots, e_n$ allows us to identify both V and V^* with $\mathbb{R}^n$ and hence to compute the standard (distributional) Fourier transform

$$\widehat{\phi}(\xi) := \int_{\mathbb{R}^n} e^{i\xi \cdot v} \phi(v) dv.$$

This transform depends on the choice of basis, or rather on the Lebesgue measure associated to it. However, the form $\widehat{\phi}\, d\xi_1 \wedge \cdots \wedge d\xi_n$ does not. Up to a constant factor, the *Fourier transform* of ϕ is the contraction of this n-form with the Euler vector field, $X_E(\xi) = \xi$, in V^*:

$$\check{\phi} := \frac{-1}{4(2\pi)^{n-1}} \widehat{\phi}\, d\xi_1 \wedge \cdots \wedge d\xi_n \rfloor X_E.$$

It is known (see [51], pages 167-168) that $\widehat{\phi}$ is smooth on $V^* \setminus \{0\}$ and homogeneous of degree $-n-1$. It follows that $\check{\phi}$ is a smooth differential form on $V^* \setminus \{0\}$ which is homogeneous of degree -1.

Definition 3.3. Let (V, φ) be an n-dimensional Minkowski space. For each integer k, $1 \leq k < n$, define the integrand

$$\varphi_k(v_1 \wedge \cdots \wedge v_k) := \int_{(\xi_1, \dots \xi_k) \in S^{*k}} |\xi_1 \wedge \cdots \wedge \xi_k \cdot v_1 \wedge \cdots \wedge v_k| \check{\varphi}^k, \tag{5}$$

where S^* is any closed hypersurface in $V^* \setminus \{0\}$ that is starshaped with respect to the origin

Theorem 3.9 ([13]). *Let (V, φ) be an n-dimensional Minkowski space. If $N \subset V$ is an immersed submanifold of dimension k, $1 \leq k < n$, then we have the following formula for the Holmes-Thompson k-area of N:*

$$vol_k(N) = \frac{1}{\epsilon_k} \int_N \varphi_k,$$

where ϵ_k denotes the volume of the Euclidean unit ball of dimension k.

To end this section we mention how to extend the definition of volume and area to general Finsler manifolds. In the case of the Busemann definition, we simply define the volume of a Finsler manifold as its Hausdorff measure. To define the Holmes-Thompson volume of a Finsler manifold (M, φ), let $D^*_m M \subset T^*_m M$ be the dual of the convex set $\{v_m \in T_m M : \varphi(v_m) < 1\} \subset T_m M$ and define the *unit co-disc bundle*, $D^* M \subset T^* M$, as the the union of all the $D^*_m M$, $m \in M$.

Definition 3.4. The *Holmes-Thompson volume* of an n-dimensional Finsler manifold (M, φ) is the symplectic volume of its unit co-disc bundle divided by the volume of the Euclidean n-dimensional unit ball. The k-volume of a k-dimensional submanifold is the volume of the submanifold with its induced Finsler metric.

Note that, using the notation of the previous section, we have that

$$vol_n(M) := \frac{1}{\epsilon_n} \int_{S^*_H M} \alpha_H \wedge (d\alpha_H)^{n-1}, \tag{6}$$

where ϵ_n is the volume of the Euclidean unit ball of dimension n.

Using the Blaschke-Santalo inequality, Durán has remarked in [40] that the Hausdorff measure of a Finsler manifold is no less than its Holmes-Thompson volume. Equality holds if and only if the metric is Riemannian.

A word on notation: From now on, the words *volume, area,* and *k-area* will relate to the Holmes-Thompson definition.

4. Unit spheres in Minkowski spaces

The simplest examples of Finsler manifolds are Minkoswski spaces and the simplest nontrivial submanifolds of Minkowski spaces are the unit spheres. However, very little is known about their intrinsic geometry. The classic results in the two-dimensional case are the theorems of Golab and Schäffer (see [44] and [58]).

Theorem 4.1 (Golab). *The length of the unit circle of a two-dimensional normed space is greater than or equal to six and less than or equal to eight. Moreover, the lower bound is attained if and only if the unit circle is a hexagon and the upper bound is attained if and only if the unit circle is a parallelogram.*

Theorem 4.2 (Schäffer). *The length of the unit circle of a two-dimensional normed space equals the length of the unit circle of its dual.*

Sketch of the proof of Golab's theorem. The key remark in the proof is that if K_1 and K_2 are two compact convex subsets of a Minkowski plane such that K_1 is contained in K_2, then the perimeter of K_1 is less than or equal to the perimeter of K_2. This can be proved elementarily or by means of integral geometry as in [10].

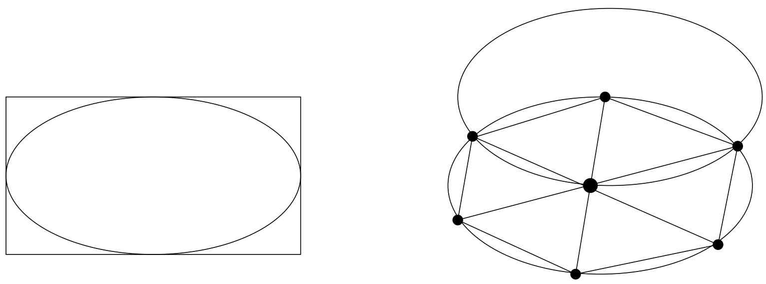

Figure 1.

Figure 1 indicates that the unit circle of a Minkowski space is circumscribed by a parallelogram of length eight and circumcribes a hexagon of length six. □

The perimeter of the unit circle is at once its 1-volume, twice its intrinsic diameter, the length of its shortest closed geodesic, and the length of its shortest closed, symmetric geodesic. Each of these interpretations points to a different, possible, higher-dimensional extension of the theorems of Golab and Schäffer. In this section, we shall quickly survey what is known and ignored about the higher-dimensional analogues of Golab's theorem. Those of Schäffer's theorem have a deep and unexpected relation with symplectic geometry and will be discussed in the next section.

We start with the following upper bounds for the Hausdorff measure and the (Holmes-Thompson) area of unit spheres in finite dimensional normed spaces.

Theorem 4.3 (Busemann and Petty). *The (intrinsic) Hausdorff measure of the unit sphere of an n-dimensional normed space is at most $2n$ times the volume of the Euclidean unit ball of dimension $n-1$. Equality holds if and only if the unit ball is a parallelotope.*

Since the Holmes-Thompson area is always less than or equal to the Hausdorff measure ([40]), we have the following corollary:

Corollary 4.1 (Thompson). *The area of the unit sphere in an n-dimensional normed space is at most $2n$ times the volume of the Euclidean unit ball of dimension $n-1$. Equality holds if anf only if $n=2$ and the unit ball is a parallelogram.*

The quest of the lower bound is much more interesting as it involves the Mahler-Reisner inequality, the Petty projection inequality, and the isosystolic inequality for Finsler metrics on projective spaces. In the three-dimensional case, the best results so far are:

Theorem 4.4 (Thompson, [66]). *The area of the unit sphere in an 3-dimensional hypermetric space or its dual is at least $24/\pi$.*

Theorem 4.5 (Álvarez, [6]). *The area of the unit sphere in a 3-dimensional normed space is at least $18/\pi$.*

Both results are far from the lowest experimental value of $36/\pi$ attained by the rhombic dodecahedron and its dual, the cubo-octahedron (see [50]). A remarkable feature of theorem 4.5 is that it holds for any reasonable notion of area. The only requirements are that isometric surfaces be assigned the same area, shorter metrics be assigned smaller areas, and Riemannian metrics be assigned their standard areas.

Problem 7 (Thompson, [65]). Find sharp bounds for the volume of the unit sphere of an n-dimensional normed space. In particular, does there exist a three-dimensional normed space such that the area of its unit sphere is less than $36/\pi$?

Schäffer has considered the length of the shortest closed geodesic which is symmetric about the origin. This length, which Schäffer calls the *girth* of the normed space is twice the the length of the shortest non-contractible geodesic, the *systole*, for the induced Finsler metric on the the projective space.

Theorem 4.6 (Schäffer, [59]). *The girth of an n-dimensional normed space is at most eight and least $4+2[n/2]^{-1}$, where $[\cdot]$ denotes the greatest-integer function. Moreover if the girth equals eight, then the space is two-dimensional and its unit ball is a parallelogram.*

In particular, the above result states that systole of the Finsler metric on $\mathbb{R}P^2$ induced by that of a unit sphere in a three-dimensional Minkowski space is at least three. This and the following isosystolic inequality imply that the area of $\mathbb{R}P^2$ with such a metric is at least $9/\pi$. The area of the unit sphere is then at least $18/\pi$ and we recover theorem 4.5.

Theorem 4.7 (Álvarez, [6]). *The area of $\mathbb{R}P^2$ with a given Finsler metric is at least $1/\pi$ times the square of the length of its shortest, closed, non-contractible geodesic.*

Problem 8. Is there a Finsler metric on the projective plane whose area is less than $2/\pi$ times the square of the length of its shortest, closed, non-contractible geodesic? If so, can such a metric be induced from that of the unit sphere of a three-dimensional Minkowski space?

Notice that a negative answer to either of these question implies that the area of the unit sphere of a three-dimensional normed space is at least $36/\pi$.

Here is yet another interesting question of Schäffer about the girth of normed spaces. The problem was posed as a conjecture in page 97 of [59].

Problem 9 (Schäffer, [59]). Prove or disprove that girth of a three-dimensional Minkowski space is at most 2π and that equality holds if and only the space is Euclidean.

Of course, one may also ask about the properties of the shortest, but not necessarily symmetric, closed geodesic on the unit sphere of a normed space.

Problem 10. Find sharp bounds for the length of the shortest closed geodesic on the unit sphere of an n-dimensional Minkowski space. Does there exist a three-dimensional Minkowski space whose unit sphere has a closed geodesic of length less than six?

Problem 11. Is the shortest closed geodesic on the unit sphere of a Minkowski space simple?

Problem 12 (Thompson). Is the shortest closed geodesic on the unit sphere of a Minkowski space symmetric with respect to the origin?

An affirmative answer to this last question would imply that the shortest closed geodesic is simple and that that the bounds in theorem 4.6 apply to the shortest closed geodesic.

Now we turn to the intrinsic diameter of unit spheres in normed spaces.

Theorem 4.8 (Schäffer, [59]). *The (intrinsic) diameter of the unit sphere of an n-dimensional normed space is at most four and at least $2 + [n/2]^{-1}$, where $[\cdot]$ denotes the greatest-integer function. In particular, the diameter of the unit sphere of a three-dimensional Minkowski space is between three and four.*

Proof. In order to see that the diameter is at most four, let x and y be any two distinct points on the unit sphere and consider a plane passing through these points and the origin. The intersection of the plane with the sphere is a curve whose length, by Golab's theorem, is at most eight. It follows that the distance between x and y is at most four.

To obtain the lower bound notice that, by theorem 4.6, the length of the shortest closed, symmetric curve on the unit sphere is greater than or equal to $4 + 2[n/2]^{-1}$.

This implies that there is a pair of antipodal points at a distance greater than or equal to $2 + [n/2]^{-1}$ and the inequality follows. □

Schäffer also characterizes those normed spaces for which the diameter of the unit sphere equals four (see theorem 9G in page 58 of [59]).

Problem 13 (Schäffer, [59])**.** Is the (inner) diameter of the unit sphere of a finite-dimensional normed space attained at a pair of antipodes?

Basically nothing is known about the geodesic flow on unit spheres of Minkowski spaces. This makes the following problem somewhat open-ended.

Problem 14. Study the integrability of the geodesic flow on the unit spheres of Minkowski spaces. If possible, find explicit examples of non-integrable unit spheres and of integrable ones with discrete isometry groups.

The last problem in this section — a variant of problem 50 in page 311 of [65] — is an attempt to generalize a result of Archimedes in Euclidean geometry stating that the area of the unit sphere in $\mathbb{R}^3$ is equal to that of any circumscribing cylinder.

Problem 15. Prove or disprove that in any three-dimensional Minkowski space the unit sphere has a circumscribing cylinder with the same area.

5. Symplectic equivalence of Finsler manifolds

In this section we study several notions of symplectic equivalence between Finsler spaces and consider the higher-dimensional generalizations of Schäffer's theorem 4.2.

Equivalence of unit co-disc bundles. The unit co-disc bundle of a Finsler manifold is an open subset of the cotangent bundle and, as such, it carries a symplectic structure. A vaguely posed, but possibly fruitful, problem is to relate the symplectic invariants of the unit co-disc bundle to the metric invariants of the Finsler manifold.

A large class of examples of Finsler manifolds with symplectomorphic unit co-disc bundles is furnished by the following result:

Theorem 5.1 (Álvarez, [7])**.** *Let $(\mathbb{R}^n, \|\cdot\|_1)$ and $(\mathbb{R}^n, \|\cdot\|_2)$ be two Minkowski spaces and let S_1 and S_2 denote their unit spheres. The unit co-disc bundle of the Finsler metric on S_1 induced by its embedding into $(\mathbb{R}^n, \|\cdot\|_2)$ is symplectomorphic to the unit co-disc bundle of the Finsler metric on S_2^* induced by its embedding into $(\mathbb{R}^n, \|\cdot\|_1^*)$.*

In particular, the unit sphere of a Minkowski space and its dual have symplectomorphic unit co-disc bundles. The following corollary — the first of our generalizations of Schäffer's theorem — predated the theorem, and was in effect its motivation.

Corollary 5.1 (Holmes and Thompson, [50])**.** *The unit sphere of a Minkowski space and its dual have the same volume.*

Theorem 5.1 can also be used to construct examples of Finsler manifolds whose unit co-disc bundles are symplectomorphic to the unit co-disc bundle of some Riemannian manifold. The following construction, a particular case of theorem 5.1, is due to A. Reznikov (unpublished) and, independently, to Álvarez (see [5]).

Let S be a centrally symmetric convex hypersurface in $\mathbb{R}^n$ and let $S^{n-1} \subset \mathbb{R}^n$ be the Euclidean unit sphere. For each point x on the sphere consider T_xS^{n-1} as an affine hyperplane of $\mathbb{R}^n$. Define a Finsler metric φ_S on S^{n-1} by requiring that the unit tangent sphere $\{v_x \in T_xS^{n-1} : \varphi_S(v_x) = 1\}$ be the boundary of the image of S under orthogonal projection onto T_xS^{n-1}.

Theorem 5.2 (Reznikov, Álvarez). *The unit co-disc bundle of (S^{n-1}, φ_S) is symplectomorphic to that of the convex hypersurface S with the Riemannian metric induced from its embedding into $\mathbb{R}^n$.*

It is natural to ask whether the unit co-disc bundle of any Finsler manifold is symplectomorphic to the unit co-disc bundle of a Riemannian manifold. I do not believe this to be the case, but I do not have a counterexample. Nevertheless, a well-known rigidity theorem of Benci and Sikorav (see [63] and [53], pp. 365) immediately implies the following result:

Theorem 5.3. *If the unit co-disc bundles of two flat Finsler tori are symplectomorphic, then the tori are isometric.*

A positive answer to the following question would imply that the unit co-disc bundle of a non-Riemannian flat torus cannot be symplectomorphic to the unit co-disc bundle of any Riemannian torus.

Problem 16. Is the nonexistence of conjugate points in Finsler tori invariant under symplectomorphisms of the (open) unit co-disc bundle?

Indeed, the Riemannian metric would have no conjugate points and would therefore, by Burago and Ivanov's solution to the Hopf conjecture, be flat. Applying theorem 5.3 we arrive at a contradiction.

Equivalence of unit co-sphere bundles.

Definition 5.1. Let M and N be two Finsler manifolds with unit co-sphere bundles S^*M and S^*N and canonical 1-forms α_M and α_N. The Finsler manifolds M and N will be said to be *alf-equivalent* if there exists a diffeomorphism

$$F : S^*M \longrightarrow S^*N$$

and a function $f : S^*M \to \mathbb{R}$ such that $F^*\alpha_N = \alpha_M + df$. If $F^*\alpha_N = \alpha_M$, we shall say that the metrics are *α-equivalent.*

Proposition 5.1. *If M and N are two alf-equivalent Finsler manifolds, then their volumes and their length spectra are equal. Moreover, if M and N are α-equivalent, then their geodesic flows are conjugate.*

Proof. To see that the volume of M equals that of N we simply use formula 6 expressing the Holmes-Thompson volume of the manifold in terms of the canonical 1-form.

The equality of the length spectra follows from formula 2, which states that the action of a leaf of the geodesic foliation equals the length of the underlying geodesic. Indeed, if $F : S^*M \longrightarrow S^*N$ is a diffeomorphism satisfying $F^*\alpha_N = \alpha_M + df$, then F maps the geodesic foliation on S^*M to the geodesic foliation on S^*N. Moreover, we see that closed leaves are taken to closed leaves with the same action.

In the case where $F^*\alpha_N = \alpha_M$, the expression for the geodesic spray as the Reeb vector field of the canonical 1-form immediately implies that the geodesic flows are conjugate. □

Problem 17 (D. Burago)**.** If a Finsler torus has no conjugate points, is it alf-equivalent to a flat one?

A large class of examples of alf-equivalent Finsler manifolds is provided by the following analogue of theorem 5.1.

Theorem 5.4 (Álvarez, [8])**.** *Let* $(\mathbb{R}^n, \|\cdot\|_1)$ *and* $(\mathbb{R}^n, \|\cdot\|_2)$ *be two Minkowski spaces and let* S_1 *and* S_2 *denote their unit spheres. The Finsler metric on* S_1 *induced by its embedding into* $(\mathbb{R}^n, \|\cdot\|_2)$ *and the Finsler metric on* S_2^* *induced by its embedding into* $(\mathbb{R}^n, \|\cdot\|_1^*)$ *are alf-equivalent. Moreover, the diffeomorphism* F *can be taken such that it takes centrally symmetric closed geodesics to centrally symmetric closed geodesics.*

As a corollary of this result we obtain our second generalization of Schäffer's theorem:

Corollary 5.2. *The length of shortest closed geodesic on the unit sphere of a Minkowski space equals the length of the shortest closed geodesic on the unit sphere its dual.*

While the length of the shortest closed geodesic on the unit sphere of a Minkowski space is an interesting invariant, it seems very hard to prove that it is continuous with respect to any of the natural topologies in the space of convex bodies. In this respect, the girth — the infimum of the lengths of all centrally symmetric, simple, closed curves — is a much better invariant. Indeed, Schäffer proved in [59], pp. 91, that the girth of a normed space is continuous with respect to the topology induced by the Banach-Mazur distance.

Schäffer also conjectured that the girth of a normed space equals the girth of its dual and proved that it is enough to consider the case of finite-dimensional normed spaces. Lighting a candle to St. Michael and another to his dragon, he goes on to give a conjectured counterexample to the conjecture. However, theorem 5.4 together with Schäffer's results implies that the conjecture is true.

Theorem 5.5 (Álvarez, [8])**.** *The girth of a normed space equals the girth of its dual.*

The preceding theorem is our third generalization of the fact that the perimeter of the unit circle of a normed plane equals the perimeter of the dual circle. Schäffer showed in [59], pp. 110, that the fourth possible generalization — that the (intrinsic) diameter of the unit sphere in a normed space equals the diameter of the dual sphere — is false. In particular, *the diameter of a Finsler manifold is not a symplectic invariant of its unit co-disc bundle.*

The notion of α-equivalence first came up in Weinstein's work on the volume of Riemannian manifolds all of whose geodesics are closed. The symplectic nature of his proofs implies that they extend unchanged for Finsler metrics.

Theorem 5.6 (Weinstein, [67])**.** *Let* $\varphi_t, t \in [0,1]$*, be a smooth family of Finsler metrics on a compact manifold* M*. If for every* t *the geodesics of the Finsler metric* φ_t *are all closed and of fixed length* L*, independent of* t*, then* (M, φ_0) *and* (M, φ_1) *are* α*-equivalent. In particular,* (M, φ_0) *and* (M, φ_1) *have the same volume.*

Weinstein's theorem follows from the fact that the manifolds of geodesics of the metrics involved are symplectomorphic. In the next paragraph we will review the natural symplectic structure on spaces of geodesics and some of their applications to integral geometry and the minimality of submanifolds in Finsler spaces.

Manifolds of geodesics. Let M be a Finsler manifold such that its space of oriented geodesics is a manifold $G(M)$. Let S^*M denote its unit co-sphere bundle and let $\pi : S^*M \to G(M)$ be the canonical projection which sends a given unit covector to the geodesic which has this covector as initial condition.

Proposition-Definition ([15] and [24]). *Let M be a Finsler manifold with manifold of geodesics $G(M)$ and let*

$$\begin{array}{ccc} S^*M & \xrightarrow{\ i\ } & T^*M \\ {\scriptstyle\pi}\big\downarrow & & \\ G(M) & & \end{array}$$

be the canonical projection onto $G(M)$ and the canonical inclusion into T^*M. If ω_M is the standard symplectic form on T^*M, then there is a unique symplectic form ω on $G(M)$ which satisfies the equation $\pi^*\omega = i^*\omega_M$.

At first sight there seem to be very few examples of Riemannian or Finsler manifolds whose space of geodesics is smooth. The following examples will convince the reader that this is not so.

Examples.

1. *Strictly convex balls and Hadamard manifolds* ([42]). Around any point x in a Finsler manifold there is an open geodesic ball with the property that the function that assigns to every point in the ball its distance from x is strictly convex. The space of geodesics in such a geodesic ball is a smooth manifold. As a result, the space of geodesics of any complete Riemannian (Finsler) metric on $\mathbb{R}^n$ with nonpositive sectional (flag) curvature is a smooth manifold.
2. *Projective Finsler metrics.* These are Finsler metrics on open, convex subsets of $\mathbb{R}P^n$ such that projective line segments are geodesics. We will review their construction in the next section.
3. *Zoll manifolds.* These are Finsler metrics all of whose geodesics are periodic with the same minimal period. A great number of Riemannian examples has been constructed by Weinstein (see [24]).

It is interesting to determine when the manifolds of geodesics of two Finsler manifolds are symplectomorphic. Here are two interesting results in this direction:

Theorem 5.7 (Ferrand, [42]). *The manifold of geodesics of an n-dimensional Hadamard manifold is symplectomorphic to the cotangent bundle of the $(n-1)$-dimensional sphere.*

Theorem 5.8 (Ono, [55]). *The manifold of geodesics of a Zoll Finsler metric on S^3 is symplectomorphic to a complex hyperquadric in $\mathbb{C}P^3$.*

It follows from these theorems that all Hadamard manifolds and all Zoll metrics on S^3 are α-equivalent among themselves.

Problem 18. Is the manifold of geodesics of a Zoll Finsler metric on the n-sphere symplectomorphic to a complex hyperquadric in $\mathbb{C}P^n$? Is the space of all Zoll Finsler metrics connected?

The study of the symplectic geometry of the space of geodesics has interesting applications to the integral geometry of Finsler manifolds. For example, the classical integral-geometric theorem of Cauchy and its extension to finite-dimensional normed spaces is a consequence of the following symplectic equivalence.

Theorem 5.9 (Álvarez, [7]). *Let $(V, \|\cdot\|)$ be a Minkowski space, and let $M \subset V$ be a smooth quadratically convex hypersurface. The unit co-disc bundle for the induced Finsler metric on M and the set of all oriented lines in V which pass through the interior of M are symplectomorphic.*

The Crofton formula for hypersurfaces of Finsler spaces, which was also announced by Chakerian in [38], follows easily from the coarea formula and symplectic reduction.

Theorem 5.10 (Álvarez, [9]). *Let M be an n-dimensional Finsler manifold with manifold of geodesics $G(M)$. If $N \subset M$ is an immersed hypersurface and if ω^{n-1} denotes the Liouville volume form on $G(M)$, then*

$$vol_{n-1}(N) = \frac{1}{2\epsilon_{n-1}} \cdot \int_{\gamma \in G(M)} \#(N \cap \gamma)|\omega^{n-1}|,$$

where ϵ_{n-1} is the volume of the Euclidean unit ball of dimension $n-1$.

Among the most interesting consequences of theorem 5.10 is the minimality of hyperplanes in projective Finsler spaces.

Theorem 5.11. *If the geodesics of a Finsler metric on $\mathbb{R}P^n$ are projective lines then projective hyperspaces are area-minimizing in their homology class.*

Proof. If $N \subset \mathbb{R}P^n$ is a hypersurface which is homologous to a projective hyperspace, then the number of points of intersection of N with a projective line is at least one. By theorem 5.10, we have that

$$\begin{aligned} vol_{n-1}(N) &= \frac{1}{2\epsilon_{n-1}} \int_{\gamma \in G(\mathbb{R}P^n)} \#(N \cap \gamma)|\omega^{n-1}| \\ &\geq \frac{1}{2\epsilon_{n-1}} \int_{G(\mathbb{R}P^n)} |\omega^{n-1}| = vol_{n-1}(\mathbb{R}P^{n-1}). \end{aligned}$$

□

This theorem suggest studying the space of all Finsler metrics on open convex subsets of $\mathbb{R}P^3$ for which planes are minimal surfaces. In the Riemannian case this problem has been studied by Bekkar and Bryant (see [21, 22, 28]). The definition below follows the one given by Bryant in the Riemannian case.

Definition 5.2. A Finsler metric φ on an open convex set $D \subset \mathbb{R}P^3$ is said to be *planal* if, for every projective plane $P \subset \mathbb{R}P^3$, the intersection $P \cap D$ is a minimal surface in the Finsler manifold (D, φ).

Problem 19. Construct and study all planal Finsler metrics on convex open subsets of $\mathbb{R}P^3$.

Problem 20. By Busemann's theorem 3.2, hyperplanes in normed spaces are area-minimizing with respect to the Hausdorff measure. Are they area-minimizing, or at least extremizing, in projective Finsler spaces?

6. Around Hilbert's fourth problem

In modern terminology, Hilbert's fourth problem asks to construct and study all Finsler metrics on open convex subsets of $\mathbb{R}P^n$ (including $\mathbb{R}P^n$ itself) such that geodesics lie on projective lines. At the root of this problem are Minkowski's work on normed spaces and the following generalization of the Cayley-Klein model of hyperbolic geometry given by Hilbert himself:

Let $D \subset \mathbb{R}^n$ be an open domain bounded by a convex hypersurace C. If x and y are two distict points on D, denote by a and b the points of intersection of C with the line determined by x and y (see figure 2), and define the distance between these points by the equation

$$d(x,y) := \frac{1}{2}\ln\left(\frac{\|y-a\|}{\|x-a\|}\frac{\|x-b\|}{\|y-b\|}\right) . \tag{7}$$

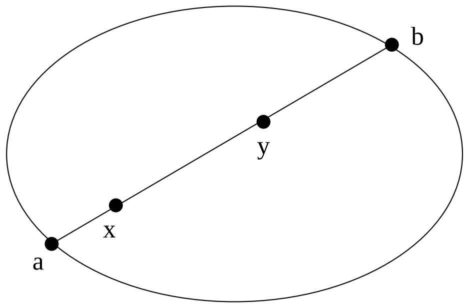

Figure 2.

Theorem 6.1 (Hilbert, [48]). *The function d is a distance function on D. Moreover, straight line segments are geodesics.*

The metric space (D, d) is called a *Hilbert geometry*. The following elegant description of the Finsler metrics that gives rise to Hilbert geometries, and which I learned from R. Ambartzumian, is apparently well known.

Let $D \subset \mathbb{R}^n$ be an open domain bounded by a smooth and quadratically convex hypersurface C. Define a Finsler metric φ on D by setting its value at a non-zero vector $v_x \in T_xD$ to be $\varphi(v_x) := (t_1^{-1} + t_2^{-1})/2$, where t_1 and t_2 are the two positive real numbers for which $x + t_1 v$ and $x - t_2 v$ belong to C.

Proposition 6.1. *If x and y are two points on D and $\overline{xy}$ is the line segment joining them, then*

$$\int_{\overline{xy}} \varphi = \frac{1}{2}\ln\left(\frac{\|y-a\|}{\|x-a\|}\frac{\|x-b\|}{\|y-b\|}\right).$$

Hamel, a student of Hilbert, was the first to study Hilbert's fourth problem. Among other things he showed that Lagrangians on $\mathbb{R}^n$ whose extremals are straight lines are characterized by a system of linear partial differential equations.

Theorem 6.2 (Hamel). *Let $\varphi : T\mathbb{R}^n \setminus 0 \to \mathbb{R}$ be a smooth Lagrangian which is homogeneous of order one in the velocities. Straight lines are extremals of the functional $\gamma \mapsto \int \varphi(\dot\gamma(t))dt$ if and only if φ satisfies the following system of equations:*

$$\frac{\partial^2\varphi}{\partial x_i \partial v_j} = \frac{\partial^2\varphi}{\partial x_j \partial v_i} \ , \ \textit{for } 1 \le i,j \le n \ . \tag{8}$$

It is mainly through the work of Busemann and Pogorelov (see [34, 57, 64, 14]) that the construction of projective Finsler metrics in terms of a class of (possibly signed) measures is now well understood.

Definition 6.1 ([14, 64]). Let $D \subset \mathbb{R}P^n$ be an open convex set and let $H_{n-1}(D)$ be the set of all hyperplanes passing through D. A possibly-signed measure on $H_{n-1}(D)$ is said to be *quasi-positive* if whenever $\overline{xy}$ and $\overline{yz}$ are two line segments not on the same line, then the measure of the set of hyperplanes intersecting twice the wedge formed by $\overline{xy}$ and $\overline{yz}$ is positive.

Theorem 6.3. *A Finsler metric φ on an open convex set $D \subset \mathbb{R}P^n$ is projective if and only if there exists a smooth quasi-positive measure Φ_{n-1} on the space of hyperplanes passing through D, $H_{n-1}(D)$, such that for any smooth curve γ*

$$\int_\gamma \varphi = \frac{1}{2}\int_{\lambda \in H_{n-1}(D)} \#(\lambda \cap \gamma)\Phi_{n-1}. \tag{9}$$

Notice that, in particular, the length of a line segment equals half the Φ_{n-1}-measure of the set of all hyperplanes intersecting it.

In Pogorelov's approach to Hilbert's fourth problem, theorem 6.3 follows from the following integral representation for the solution of Hamel's equations.

Theorem 6.4. *A Lagrangian $\varphi : T\mathbb{R}^n \setminus 0 \to \mathbb{R}$ which is homogeneous of order one in the velocities satisfies Hamel's equations if and only if there exists a smooth even function $\nu(r, \xi)$ on $\mathbb{R} \times S^{n-1}$ such that*

$$\varphi(x, v) = \frac{1}{4}\int_{\xi \in S^{n-1}} |\xi \cdot v| \nu(\xi \cdot x, \xi)\Omega \ , \tag{10}$$

where Ω is the standard area form on the unit sphere in $\mathbb{R}^n$.

Example ([12]). Applying formula 10 to the function $\nu : \mathbb{R} \times S^1 \to \mathbb{R}$ defined by $\nu(r, \theta) := 1 + r^2$, we obtain the Finsler metric

$$\varphi(x_1, x_2, v_1, v_2) = \frac{1}{3\sqrt{v_1^2 + v_2^2}}\,(2x_1x_2v_1v_2 + (3 + 2x_1^2 + x_2^2)v_1^2 + (3 + 2x_2^2 + x_1^2)v_2^2).$$

A useful consequence of equation 10 is that the convex linear combination of projective Finsler metrics is again a projective Finsler metric.

Theorem 6.3 states that projective Finsler metrics are exactly those Finsler spaces for which there is a Crofton formula for the lengths of curves. Do the Crofton formulas for the areas of submanifolds also hold? Does the first Crofton formula imply all others? The answer is *yes* if by area we mean Holmes-Thompson area:

Theorem 6.5 (Álvarez and Fernandes, [12]). *Let φ be a projective Finsler metric on an open convex domain $D \subset \mathbb{R}P^n$ and let k, $1 \le k \le n-1$, be a natural number. There exists a smooth (possibly signed) measure Φ_{n-k} on the manifold $H_{n-k}(D)$ of $(n-k)$-flats passing through D such that if $N \subset \mathbb{R}^n$ is an immersed submanifold of dimension k, then*

$$vol_k(N) = \frac{1}{\epsilon_k}\int_{\lambda \in H_{n-k}(D)} \#(N \cap \lambda)\Phi_{n-k}, \tag{11}$$

where ϵ_k is the volume of the Euclidean unit ball of dimension k.

The construction of the measures Φ_{n-k} given in [12] and the formula 5 for the Holmes-Thompson volume imply that all the tangent spaces of a projective Finsler metric are hypermetric if and only if the measures Φ_{n-k}, $k = 1, \dots, n-1$, are positive. We have then the following extension of theorem 5.11:

Theorem 6.6 (Alvarez and Fernandes). *If the geodesics of a Finsler metric on $\mathbb{R}P^n$ are projective lines and all its tangent spaces are hypermetric, then the projective subspaces are area-minimizing in their homology class.*

Proof. If $N \subset \mathbb{R}P^n$ is a k-dimensional submanifold which is homologous to a projective subspace, then the number of points of intersection of N with a projective subspace of complementary dimension is at least one. Using the Crofton formula 11 and positivity of the measure Φ_{n-k}, we have that

$$vol_k(N) = \frac{1}{\epsilon_k} \int_{\lambda \in H_{n-k}(\mathbb{R}P^n)} \#(N \cap \lambda)\Phi_{n-k} \geq \frac{1}{\epsilon_k} \int_{H_{n-k}(\mathbb{R}P^n)} \Phi_{n-k} = vol_k(\mathbb{R}P^k).$$

□

A yet unpublished result of Álvarez and Fernandes states that the projective subspaces of an arbitrary projective Finsler space are minimal submanifolds in the sense that they extremize the area functional. They could perhaps fail to be area-minimizing.

The two previous theorems point at the similarity between the standard Riemannian metric and an arbitrary projective metric on projective space. Here is a third result in this direction:

Proposition 6.2. *If $(\mathbb{R}P^n, \varphi)$ is a projective Finsler space for which the length of the projective lines is equal to π, then the Holmes-Thompson volume of $(\mathbb{R}P^n, \varphi)$ equals the volume of $\mathbb{R}P^n$ with its standard Riemannian metric.*

Proof. If φ_0 denotes the standard Riemannian metric on $\mathbb{R}P^n$, then, for each number t, $t \in [0,1]$, the metric $\varphi_t = (1-t)\varphi_0 + t\varphi$ is a projective Finsler metric and the length of its closed geodesics is π. Applying Weinstein's theorem 5.6, we conclude that the Holmes-Thompson volumes of $(\mathbb{R}P^n, \varphi_0)$ and $(\mathbb{R}P^n, \varphi)$ are equal. □

Since the Holmes-Thompson volume of a Finsler manifolds is always less than or equal to its Hausdorff measure, with equality holding only for Riemannian manifolds, we have the following:

Corollary 6.1. *If $(\mathbb{R}P^n, \varphi)$ is a projective Finsler space for which the length of the projective lines is equal to π, then its Hausdorff measure is greater than or equal to the Hausdorff measure of $\mathbb{R}P^n$ with its standard Riemannian metric. Equality holds if and only if φ is the standard Riemannian metric on $\mathbb{R}P^n$.*

The nonexistence of totally geodesic hypersurfaces, and the implied failure of the integral geometric construction, may be the reason why the analogues of Hilbert's fourth problem for rank-one symmetric spaces other than $\mathbb{R}P^n$ has never been studied. On the other hand, this makes the problem more interesting.

Problem 21. Construct all Finsler metrics on $\mathbb{C}P^n$ such that the geodesics coincide as point sets with those of the standard Riemannian metric on $\mathbb{C}P^n$.

In trying to solve this problem, Álvarez and Durán stumbled upon the following construction of Finsler metrics on $\mathbb{C}P^n$ for which the complex projective lines are totally geodesic and the geodesics are circles.

Construct a Finsler metric on S^{2n+1} all of whose geodesics are great circles by first constructing a Finsler metric ϕ on $\mathbb{R}P^{2n+1}$ whose geodesics are projective lines and then lifting it to the sphere by the canonical covering map from S^{2n+1} to $\mathbb{R}P^{2n+1}$. This metric can be constructed in such a way that the circle action of the Hopf fibration acts by isometries: just choose a smooth volume density in $\mathbb{R}P^{2n+1*}$ which is invariant with respect to the induced action.

Define the Finsler metric φ on $\mathbb{C}P^n$ by defining the hypersurfaces $S_z := \{v_z \in T_z\mathbb{C}P^n : \varphi(v_z) = 1\}$ as follows : take a point $x \in S^{2n+1}$ with $\rho(x) = z$ and let S_z be the boundary of the image of the hypersurface $\{w_x \in T_xS^{2n+1} : \phi(w_x) = 1\}$ under the linear projection $D\rho : T_xS^{2n+1} \to T_z\mathbb{C}P^n$. The symmetry of the metric ϕ implies that S_z does not depend on the choice of the point x with $\rho(x) = z$.

Theorem 6.7 (Álvarez and Durán, [11]). *The geodesics of the Finsler metric φ are circles. In particular, complex projective lines are totally geodesic.*

Problem 22. Construct all Finsler metrics on $\mathbb{C}P^n$ such that the geodesics are circles.

There have been many attempts to define the Finsler analogue of Kahler metrics, but none seems to have enjoyed any measure of success. Since the metric properties of Kahler manifolds are not so well understood as to proceed in this direction, it makes sense to use some remarkable geometric property of Kahler manifolds in an attempt to define their Finsler analogues. One such remarkable geometric property is that complex submanifolds are minimal.

Problem 23. Construct and study all the Finsler metrics on $\mathbb{C}P^n$ for which complex submanifolds are minimal.

7. Closed geodesics

In Riemannian geometry, the study of closed geodesics has a long and glorious history with contributors like Poincare, Birkhoff, and Morse. However, there do not seem to be many interesting results about closed geodesics in Finsler manifolds. If an existence result in Riemannian geometry depends only on Morse theory and the method of broken geodesics, then it holds in Finsler geometry.

A Riemannian result that would be interesting to extend to the Finsler setting is a theorem of Bangert and Franks ([17, 43]) stating that any Riemannian metric on the two-dimensional sphere has infinitely many closed geodesics. Here we emphasize that we're considering symmetric or reversible Finsler metrics. Indeed, Katok has constructed nonsymmetric Finsler metrics on the 2-sphere which have only two closed geodesics (see [52, 68]).

Problem 24 (Bangert). Does every Finsler metric on S^2 have infinitely many closed geodesics?

Franks' work on the periodic points of area preserving maps of the annulus and recent work by Hofer, Wysocki, and Zehnder (see [49]) suggest proposing the following conjecture

Conjecture. *The number of distinct closed geodesics on a not necessarily symmetric Finsler metric on the 2-sphere is either two or infinity.*

8. Differential invariants of Finsler surfaces

In this section we follow the lecture notes [10] in defining the differential invariants of Finsler surfaces without the aid of Cartan's connection and then introducing Cartan's structure equations in order to get at the deep relations between these invariants.

8.1. Convex geometry and the invariant I. A smooth, centrally symmetric, and quadratically convex curve S on a two-dimensional vector space V parametrizes a family of Euclidean structures on V. Indeed, for each point $v \in S$, there is a unique ellipse $E(v)$ which is centered at the origin and osculates S up to second order at this point. We associate to v the Euclidean structure with $E(v)$ as unit circle.

Definition 8.1. Let (V, φ) be a two-dimensional Minkowski space with unit circle S. An orthonormal basis of V is a pair of vectors $v, v^{\perp}$, where $v \in S$ and $v^{\perp}$ is both of unit length and perpendicular to v with respect to the Euclidean structure associated to v.

Using the Euclidean structures associated to the curve we may define the *distinguished parametrization of* S: orient the vector space V and parametrize the curve S by a map γ in such a way that $\{\gamma(t), \dot{\gamma}(t)\}$ is an oriented orthonormal basis. Tabachnikov has remarked (private communication) that this parametrization is the only one satisfying the equation

$$\det(\gamma(t), \dot{\gamma}(t)) = \det(\dot{\gamma}(t), \ddot{\gamma}(t)).$$

Using the distinguished parametrization, we now define a centro-affine invariant of the curve S.

Definition 8.2. The period of the curve γ will be called the *total angle* of S.

While it is obvious that the total angle of an ellipse equals 2π, the following geometric inequality comes as a complete surprise.

Theorem 8.1 (Schneider, [60])**.** *The total angle of a smooth, centrally-symmetric, and quadratically convex curve on the plane is less than or equal to 2π. Equality holds if and only if the curve is an ellipse.*

Problem 25. Is there any centrally symmetric convex surface in $\mathbb{R}^3$ which is not an ellipsoid and such that the total angles of all of its sections with 2-dimensional subspaces are equal ?

We shall see the significance of this problem for Finsler geometry after we talk about Landsberg surfaces and Chern-Bao's version of the Gauss-Bonnet theorem for Finsler surfaces. Before we do this, we shall define another centro-affine invariant of S which measures how the Euclidean structures associated to the points of S vary. Again, we make use of the distinguished parametrization of S.

Definition 8.3. Let S be a smooth, centrally symmetric, and quadratically convex curve on a two-dimensional vector space V. If γ is the distinguished parametrization of S and $v = \gamma(t)$ is a point on S, write $\dddot{\gamma}(t)$ in the basis $\{\gamma(t), \dot{\gamma}(t)\}$ and define $I(v)$ as the the component of $\dddot{\gamma}(t)$ in the direction of $\dot{\gamma}(t)$.

Note that $I(v) = 0$ if and only if the osculating ellipse $E(v)$ osculates S up to third order or higher at v. It follows that if I is identically zero, then S is an ellipse. Using a theorem of Ghys of the zeros of the Schwarzian derivative (see [56]), Álvarez has proved the following global property of I:

Proposition 8.1 ([10]). *For any smooth, centrally symmetric, and quadratically convex curve, the invariant I vanishes at least eight times.*

If (M, φ) is a Finsler surfaces, then every tangent space $T_m M, m \in M$, is a Minkowski space and the indicatrix

$$S_m M := \{v_m \in T_m M : \varphi(v_m) = 1\}$$

is smooth, centrally symmetric, and quadratically convex. For each unit tangent vector v_m we define $I(v_m)$ as the value of the invariant I of $S_m M$ at the point v_m. The result is a smooth function, which we again denote by I, defined of the unit circle bundle of M. Clearly, this function is identically zero if and only if the Finsler surface is Riemannian.

We can also use the previous geometric constructions to define a vector field on the unit circle bundle of an oriented Finsler surface (M, φ): give every unit tangent circle its distinguished parametrization and let the vector field X_3 be the velocity field. This vector field is of great importance in Cartan's approach to Finsler surfaces.

8.2. The invariant J. We shall now define a smooth function on the unit bundle of M which measures how the invariant I changes along the geodesics.

Definition 8.4. Let (M, φ) be a Finsler surfaces, let $v_m \in T_m M$ be a unit tangent vector, and let $\gamma : (-\epsilon, \epsilon) \longrightarrow M$ be the geodesic with initial condition v_m. Define

$$J(v_m) = d/dt \; I(\dot{\gamma}(t))_{|t=0}.$$

Note that $J = 0$ means that I is an invariant of motion.

Definition 8.5. A Finsler surface for which the invariant J is identically zero is called a *Landsberg surface* .

Unfortunately, all the known examples of Landsberg surfaces are locally isometric to Minkowski planes. For example, the following problem remains open:

Problem 26. Is there any Landsberg metric on the 2-sphere?

Problem 27. Does there exist a non-Riemannian, compact Landsberg surface that can be isometrically embedded in some Minkowski 3-space?

In view of the following proposition, a negative answer to problem 25 implies a negative answer to the preceding one.

Proposition 8.2. *Let (M, φ) be a Landsberg surface. If M is connected, then the total angle of any two of its tangent unit circles is the same.*

8.3. Curvature of Finsler surfaces. The formula of second variation really belongs to variational calculus and not to Riemannian geometry. It should not come then as a surprise that many of the definitions and theorems in Riemannian geometry, including the notion of curvature, extend to Finsler geometry.

Let us start by defining the obvious extensions of Jacobi fields and conjugate points:

Definition 8.6. Given a geodesic $\gamma : [a, b] \to M$, a *geodesic variation* of γ is a smooth map $\Gamma : (-\epsilon, \epsilon) \times [a, b] \to M$ such that

- $\Gamma(0, t) = \gamma(t)$.
- For each fixed s_0, the curve $t \mapsto \Gamma(s_0, t)$ is a geodesic.

Definition 8.7. Let $\Gamma(s,t)$ be a geodesic variation of γ. The vector field Y along γ defined by

$$Y(t) = \frac{\partial \Gamma(s,t)}{\partial s}|_{s=0}$$

is called a *Jacobi field*. A Jacobi field is said to be *proper* if, for each t, $\dot\gamma(t)$ and $Y(t)$ form an orthonormal basis of the Minkowski plane $T_m M$.

Definition 8.8. Let p be a point in a Finsler surface and let γ be a geodesic starting from p. A point $\gamma(s)$ is said to be *conjugate* to p along γ if there exists a nonzero Jacobi field $Y(t)$ along γ such that $Y(0) = Y(s) = 0$.

Just as in Riemannian geometry, geodesics minimize up to their first conjugate point.

Proposition 8.3. *Let p be a point in a Finsler surface and let γ be a geodesic starting from p. The curve γ restricted to the interval $[0,s]$ minimizes length if and only if there is no conjugate point between $p = \gamma(0)$ and $\gamma(s)$.*

If we are given a geodesic γ let us orient the tangent spaces $T_{\gamma(t)}M$ and let us define $\dot\gamma^{\perp}(t)$ in such a way that $\dot\gamma(t)$ and $\dot\gamma^{\perp}(t)$ form an oriented orthonormal basis on $T_{\gamma(t)}M$. Note that any proper Jacobi field Y along γ can be uniquely expressed as

$$Y(t) = y(t)\dot\gamma^{\perp}(t),$$

where y is a real-valued function.

Proposition-Definition 8.1. There is a unique smooth function K on the unit circle bundle of M such that for any geodesic γ parametrized with unit speed and for any proper Jacobi field $Y(t) = y(t)\dot\gamma^{\perp}(t)$ the *Jacobi equation*

$$y''(t) + K(\dot\gamma(t))y(t) = 0$$

holds.

The function K is called the *curvature* of (M,φ). In contrast with the Riemannian case, K depends on both the point $\gamma(t) \in M$ and the direction of $\dot\gamma(t)$.

The Bonnet-Myers theorem follows as usual from the Jacobi equation and Sturm's comparison theorem.

Theorem 8.2. *If the curvature function of a Finsler surface (M,φ) is greater than or equal to a positive number δ, then the diameter of M is less than or equal to $\pi/\sqrt{\delta}$.*

We end by listing of some of the things that are known, and ignored, about Finsler surfaces of constant curvature.

Theorem 8.3 (Funk). *The Hilbert geometries have constant curvature equal to -1.*

Theorem 8.4 (Akbar-Zadeh, [4]). *A compact Finsler surface with zero curvature is locally isometric to a Minkowski plane.*

Theorem 8.5 (Akbar-Zadeh, [4]). *A compact Finsler surface of constant negative curvature is Riemannian.*

Problem 28 (Bryant, [26]). Construct a Finsler metric on the sphere with constant positive curvature.

8.4. Cartan's Structure Equations. By now we have defined defined three geometric invariants of Finsler surfaces: I, J and K. The invariant I is a centro-affine invariant which descibes the shape of each unit tangent circle, the invariant K belongs to the calculus of variations and measures the focusing of geodesics, and the invariant J, by measuring how I varies along geodesics, partakes of both convex geometry and variational calculus. All three invariants can be defined, as we have done, by elementary geometric and variational considerations, but there is nothing to suggest the deep and interesting relations between the three.

Definition 8.9. If (M, φ) is a Finsler manifold, the *geodesic spray* of M is the vector field X_1 defined on the unit tangent bundle and whose value at a unit tangent vector v_m is defined as follows: take $\gamma(t)$ to be the geodesic with initial condition v_m and set $X_1(v_m) := d/dt\ \dot{\gamma}(t)_{|t=0}$.

Theorem 8.6 (Cartan, [37]). *Let (M, φ) be an oriented Finsler surface, let X_1 denote its geodesic spray, and let X_3 be the vector field defined at the end of 8.1. If we define $X_2 := [X_3, X_1]$, then we have the following equations:*

$$\begin{aligned} [X_3, X_1] &= X_2, \\ [X_1, X_2] &= KX_3, \\ [X_3, X_2] &= -X_1 + IX_2 + JX_3. \end{aligned}$$

Of course, Cartan prefered differential forms to vector fields and he wrote the above equations in terms of the dual forms ω_1, ω_2 and ω_3 defined by the equations $\omega_i(X_j) = \delta_{ij}$. Cartan's *structure equations* are:

$$\begin{aligned} d\omega_1 &= -\omega_2 \wedge \omega_3, \\ d\omega_2 &= \omega_1 \wedge \omega_3 - I\omega_2 \wedge \omega_3, \\ d\omega_3 &= -K\omega_1 \wedge \omega_2 - J\omega_2 \wedge \omega_3. \end{aligned}$$

Note that by differentiating the structure equations we obtain the following *Bianchi identities*:

$$\begin{aligned} J &= I_1, \\ K_3 + KI + J_1 &= 0. \end{aligned}$$

In these identities the subindices represent differentiation with respect to the vector fields X_1, X_2 and X_3. In general if F is a function on the unit bundle we will write

$$dF = F_1\omega_1 + F_2\omega_2 + F_3\omega_3\ .$$

Cartan shows that the forms ω_1, ω_2, and ω_3 solve the problem of equivalence. From this it follows, at least in theory, that all local invariant properties of Finsler surfaces can be written in terms of the functions I, J, K, and their derivatives with respect to the vector fields X_1, X_2, and X_3. As an example, we have Berwald's characterizations of locally Minkowski and projectively flat Finsler surfaces.

Theorem 8.7 (Berwald). *A Finsler surface is locally isometric to a Minkowski plane if and only if K, J, and I_2 are identically zero.*

Problem 29. Is there any complete Finsler metric on $\mathbb{R}^2$ with zero curvature, but which is not locally Minkowski? It may be shown that such a metric cannot be obtained from an embedding of $\mathbb{R}^2$ into any Minkowski space.

Definition 8.10. A Finsler manifold is said to be *projectively flat* if around every point we can find a small neighbourhood and a diffeomorphism of this neighbourhood to an open subset of Euclidean space such that geodesics are mapped onto straight lines.

Theorem 8.8 (Berwald, [23]). *The geodesics of a Finsler surface are locally the geodesics of some affine connection if and only if the following equation holds:*

$$I_{23} + J_{33} + 2I(I_2 + J_3) + 6J = 0.$$

The dual system of curves to the geodesics of a Finsler surface are locally the geodesics of some affine connection if and only if the following equation holds:

$$K_{31} - 3K_2 = 0$$

Moreover, both of the above equations hold if and only if the surface is projectively flat.

For dual systems of curves and an elementary exposition of path geometry see pages 42-56 of Arnold's book [16]. A very clear exposition of Berwald's theorem and its proof is given in [27].

It is amusing to rederive the following classic theorem of Beltrami from Berwald's result.

Corollary 8.1 (Beltrami). *A Riemannian surface is projectively flat if and only if its curvature is constant.*

8.5. Applications of Cartan's structure equations. Armed with Cartan's structure equations and their Bianchi identities, we are now in a position to prove some of the deepest results in the theory of Finsler surfaces. We start by theorem 8.5: *If (M, φ) is a compact Finsler surface of constant negative curvature, then M is Riemannian.*

Proof. Using the second Bianchi identity, which tells us that $K_3 + KI + J_1 = 0$, we have that if $K \equiv c$ is a constant and $\gamma(t)$ is a geodesic on M, then the function $I(t) := I(\dot{\gamma}(t))$ satisfies the differential equation

$$\frac{d^2}{dt^2} I = -cI \ .$$

If c is negative, then $I(t)$ must be a linear combination of exponentials and, therefore, if the initial condition is not $I(0) = 0, I'(0) = 0$, the function $I(t)$ is unbounded. Since I is bounded whenever the Finsler surface is compact, the only possibility that remains is that I be identically zero, and that the surface be Riemannian. □

Shen has remarked in [62] that if the invariant I is unbounded, then the surface does not admit an isometric embedding into any Minkowski space (this follows at once from the geometric interpretation of I given in 8.1). As a consequence of this remark, the above result, and theorem 8.3, we have the following simple proposition:

Proposition 8.4 (Álvarez and Durán, [10]). *The Hilbert geometry given by a convex curve C does not admit an isometric embedding into a Minkowski space unless C is an ellipse.*

If C is an ellipse, then the Hilbert geometry is the Cayley-Klein model of hyperbolic geometry and, by a theorem of Rosendorn, it admits an explicit isometric embedding into $\mathbb{R}^5$ (see page 276 in [46]).

All the previous applications involve only the Bianchi equations and not the structure equations *per se*. An application that does is the proof — taken from [26] — of proposition 8.2: *Let (M, φ) be a Landsberg surface. If M is connected, then the total angle of any two of its tangent unit circles is the same.*

Proof. If x and y be two points on M, the difference of the total angle of the unit circle S_y over y and the total angle of the unit circle S_x over x is given by

$$\int_{S_y} \omega_3 - \int_{S_x} \omega_3 \ ,$$

where the orientation over S_y and S_x is taken so that the integrals are positive.

Let $\gamma : [0,1] \to M$ be a smooth curve joining them. Let $\pi : UM \to M$ denote the natural projection and set $\mathcal{C}$ be the cylinder $\pi^{-1}(\gamma)$. Note that the oriented boundary of $\mathcal{C}$ is $S_y - S_x$ and that the 2-form $\omega_1 \wedge \omega_2$ vanishes identically on $\mathcal{C}$.

Using Stokes theorem and the structure equations we have that

$$\int_{S_y} \omega_3 - \int_{S_x} \omega_3 = \int_{\mathcal{C}} d\omega_3 = \int_{\mathcal{C}} -K\omega_1 \wedge \omega_2 - J\omega_2 \wedge \omega_3 = 0.$$

□

Another application that uses the full power of Cartan's structure equations is the Finsler version of the Gauss-Bonnet theorem given by Bao and Chern in [18].

Let (M, φ) be a compact, oriented Finsler surface and let X be a vector field on M with a finite number of nondegenerate zeros. Cut out small discs, say of radius r, around the zeros of X and denote the resulting manifold with boundary by M_r. Normalizing the vector field X on M_r we obtain a section $\sigma_r : M_r \to SM_r$ over the unit circle bundle of M_r.

Using the fact that the total angle of the tangent unit circle at a point x is given by the integral of ω_3 over S_xM, the equation

$$d\omega_3 = -K\omega_1 \wedge \omega_2 - J\omega_2 \wedge \omega_3, \tag{12}$$

and Stokes' theorem, Chern and Bao arive at the following result:

Theorem 8.9 ([18]). *Let (M, φ) be a compact, oriented Finsler surface and let X be a vector field on M with zeros $x_1, \dots, x_n$ which are nondegenerate and with indices $\mathcal{I}(x_1), \dots \mathcal{I}(x_n)$. Using the notation above, the limit as r tends to zero of the quantity*

$$\int_{M_r} \sigma_r^*(-K\omega_1 \wedge \omega_2 - J\omega_2 \wedge \omega_3) \tag{13}$$

is well defined and equals $\sum_{i=1}^n \mathcal{I}(x_i)\mathcal{A}(x_i)$, where $\mathcal{A}(x_1), \dots, \mathcal{A}(x_n)$ are the total angles of the unit tangent circles at the points $x_1, \dots, x_n$.

If, as in the case of Landsberg surfaces, the total angle of the unit tangent circles does not vary from point to point, then we have that the Euler characteristic of M can be written as an integral in terms of the differential invariants of the Finsler surface.

Problem 30. Find explicit, concrete examples of Finsler metrics on the sphere where the total angle of the unit tangent circles does not vary from point to point

In Riemannian geometry, the Gauss-Bonnet theorem has many interesting geometrical consequences. For example, it can be used to show that if the absolute value of the curvature of a Riemannian 2-sphere is less than or equal to one, then its area is at least 4π. Moreover, equality holds if and only if the curvature is constantly equal to one. The Finsler version of the Gauss-Bonnet theorem does not seem to have any geometric significance, but we may still expect that small curvature implies large volume and that large curvature implies small volume. Surprisingly, there do not seem to exist any theorems to this effect.

Problem 31. What is the infimum of the areas of Finsler 2-spheres with $|K| \leq 1$?

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J.C. Álvarez Paiva, Université Catholique de Louvain, Institut de Mathématique Pure et Appl., Chemin du Cyclotron 2, B–1348 Louvain–la–Neuve, Belgium.

E-mail address: alvarez@agel.ucl.ac.be