# MINIMAL ENTROPY RIGIDITY FOR FINSLER MANIFOLDS OF NEGATIVE FLAG CURVATURE

JEFF BOLAND AND FLORENCE NEWBERGER

Abstract. We define a normalized entropy functional for compact Finsler manifolds of negative flag curvature. Using the method of Besson, Courtois, and Gallot, we show that among all such manifolds that are homotopy equivalent to a compact, Riemannian, locally symmetric manifold of negative curvature, the entropy functional is minimized precisely on the locally symmetric manifold.

## 1. Introduction

In 1982, A. Katok [10] proved results implying that among all negatively curved Riemannian metrics on a compact, locally symmetric manifold which are conformally equivalent to the locally symmetric metric and of the same volume, the locally symmetric metric has the absolute smallest volume growth entropy. In dimension 2, it follows that a constant curvature metric on a surface of higher genus is the absolute minimum for volume growth entropy among all other negatively curved metrics of the same volume. This result was extended in 1995 to Riemannian manifolds, not necessarily of negative curvature, for which there exists a map of non-zero degree to a higher-dimensional rank one locally symmetric manifold by Besson, Courtois, and Gallot [3]. To state the version of their theorem relevant for us, let $(X, g_o)$ be a compact, negatively curved, locally symmetric manifold, and let $g$ be any negatively curved metric on a compact Riemannian manifold $Y$ homotopy equivalent to $X$. The volume growth entropy of the metric $g$, denoted by $h(g)$, is the asymptotic exponential growth rate of balls in the universal cover

$$h(g) = \lim_{R\to\infty} \frac{1}{R} \log(\mathrm{Vol} B(y, R)),$$

for any $y \in \tilde{Y}$ and $B(y, R)$ the ball of radius $R$ in $\tilde{Y}$ about $y$ with respect to $g$. The normalized entropy functional of a negatively curved, compact, Riemannian manifold $(Y^n, g)$ is the quantity $h(g)^n \mathrm{Vol}(Y, g)$ and with this notation the minimal entropy rigidity theorem is:

**Theorem 1** (Besson-Courtois-Gallot [3], [4]). *Let $(X, g_o)$ be a compact, $n$-dimensional, Riemannian, locally symmetric manifold of negative curvature ($n \geq 3$), and $(Y, g)$ a compact, negatively curved, Riemannian manifold homotopy equivalent to $(X, g_o)$. Then*

*(i)* $h(g_o)^n \mathrm{Vol}(X, g_o) \leq h(g)^n \mathrm{Vol}(Y, g)$ *and*
*(ii)* $h(g)^n \mathrm{Vol}(Y, g) = h(g_o)^n \mathrm{Vol}(X, g_o)$ *implies that* $(Y, g)$ *is homothetic to* $(X, g_o)$.

---

*Date*: September 23, 1999.
1991 *Mathematics Subject Classification.* Primary 53C20, Secondary 53B40, 53C35.

In [4], the authors conjecture that Theorem 1 remains true in the class of Finsler metrics. In this paper we define a normalized entropy functional on the space of compact, reversible, Finsler manifolds of negative flag curvature that are homotopy equivalent to a compact, Riemannian, locally symmetric manifold $(X, g_o)$ of negative curvature, and show that $(X, g_o)$ is the unique minimum for this functional.

To state our theorem, we briefly recall some notions from Finsler geometry; see [13] or [1] and the references therein for more details. A *Finsler structure* on a compact manifold $Y$ is a function $F : TY \to \mathbb{R}^+$ satisfying:

(a) $F(y, tv) = tF(y, v)$ for all $t \geq 0$,
(b) $F^2$ is strictly positive and $C^\infty$ on $TY$ minus the zero section, and
(c) in local coordinates $(y_i, \dot{y}_i)$ on $TY$, the matrix of partial derivatives $\frac{\partial^2 F^2}{\partial \dot{y}_i \partial \dot{y}_j}$ is positive definite.

$F$ induces a distance function on $Y$ for which $(Y, F)$ is a geodesic metric space. Note that any Riemannian manifold $(Y, g)$ is a Finsler manifold with $F = \|\cdot\|_g$. We say $F$ is Riemannian if $F = \|\cdot\|_g$ for some Riemannian metric $g$, and $F(y, \cdot)$ is Euclidean if it is the norm induced on $T_yY$ by an inner product.

The unit ball for $F$ in each tangent space is a smooth and strictly convex set. A *reversible* Finsler structure is one such that the unit ball is radially symmetric, i.e. $F(-v) = -F(v)$, and in this paper we will assume our Finsler manifolds are reversible (in order to apply the work of D. Egloff [8], [9]).

Every Finsler manifold comes with a natural volume form, which we now describe. Fix an arbitrary Riemannian metric $g$ on $Y$ and let $dg$ be its volume form. Denote by $B_y(1, g)$ and $B_y(1, F)$ the unit balls of radius 1 in $T_yY$ with respect to $g$ and $F$ respectively, and let $C_n = \mathrm{Vol}_g(B_y(1, g))$ and $\mathrm{Vol}_g(B_y(1, F))$ be their volumes with respect to $g$. The Finsler volume form is given by

$$dF(y) = \frac{\mathrm{Vol}_g(B_y(1, g))}{\mathrm{Vol}_g(B_y(1, F))}\, dg = \frac{C_n}{\mathrm{Vol}_g(B_y(1, F))}\, dg,$$

which is independent of the choice of the Riemannian metric $g$.

The Finsler structure allows us to define for each nonzero $u \in T_yY$ an inner product $g_u$ on $T_yY$ in the following way. Given two vectors $v, w \in T_yY$, set

$$g_u(v, w) = \sum_{i,j=1}^{n} (g_u)_{ij} v_i w_j, \quad \text{where } (g_u)_{ij} = \frac{1}{2} \frac{\partial^2 F^2}{\partial \dot{y}_i \partial \dot{y}_j}(y, u).$$

One can check that these inner products only depend on the direction of $u$, i.e. that $g_{tu} = g_u$ for $t \neq 0$. For $u \in T_yY$, we define the *distortion* $\rho(u)$ of $F$ in the direction $u$ to be

$$\rho(u) = \frac{C_n}{\mathrm{Vol}_{g_u}(B_y(1, F))}.$$

If $F = \|\cdot\|_g$ is Riemannian then the inner products $g_u$ all equal $g$, and the distortion is 1 for each $u$. The following proposition, which we will use in the equality case of our main theorem, shows that the distortion is a measure of how far a Finsler norm is from being Riemannian. Since to our knowledge it has not appeared in print, we provide a proof due to Z. Shen.

**Proposition 2.** *$\rho(u)$ is independent of $u \in T_yY$ if and only if $F_y$ is Euclidean.*

*Proof.* For this proof only we write $g(u)$ instead of $g_u$. Introduce local coordinates around $y$ and let $V$ be the volume of $B_y(1, F)$ with respect to the volume element in those coordinates. Then letting $\Delta(u) = \det(g_{ij}(u))$, $\rho(u) = \frac{C_n}{V\sqrt{\Delta(u)}}$, so $\rho(u)$ is constant in $u$ if and only $\Delta(u)$ is constant in $u$. Let

$$A_{ijk}(u) = \frac{1}{4}F(u)[F^2]_{\dot{y}^i\dot{y}^j\dot{y}^k}(u) = \frac{1}{2}\frac{\partial g_{ij}}{\partial \dot{y}^k}(u),$$

and set $(g^{ij}(u)) = (g_{ij}(u))^{-1}$ and $A_i(u) = g^{jk}(u)A_{ijk}(u)$. $A$ is called the Cartan tensor, and $F_y$ is Euclidean if and only if $A_{ijk} \equiv 0$. Since

$$\frac{\partial}{\partial \dot{y}^i}\Delta(u) = \frac{2}{F(u)}\sum_{jk} A_{jki}(u)g^{jk}(u)\Delta(u) = \frac{2}{F(u)}A_i(u)\Delta(u),$$

$\Delta(u)$ is constant in $u$ if and only if $A_i(u) = 0$. The following result finishes it up.

**Theorem 3** (Deicke [7], [6]). *$F_y$ is Euclidean if and only if $A_i = 0$.*

□

Finally, following [8], we define the *volume growth entropy* $h(F)$ for a compact Finsler manifold exactly as in the Riemannian case, where the volume of a ball in the universal cover is computed with respect to the lift of the volume form $dF$. We also briefly give a geometric description of *flag curvature*, the Finsler generalization of Riemannian sectional curvature. Given a "flagpole" $u \in T_yY$, extend $u$ to a vector field $z \mapsto U(z)$ locally near $y$ which is tangent to geodesics. The flag curvature of a "flag" (i.e. a 2-plane) in $T_yY$ is the Riemannian sectional curvature of the 2-plane with respect to the Riemannian metric defined locally near $y$ by $z \mapsto g_{U(z)}$. The usual definition of flag curvature involves a connection on the pull-back $\pi^*TY$ of the tangent bundle and may be found in [13]; for its equivalence with the present definition, see [14]. We do not here require any detailed knowledge of flag curvature, except that Riemannian manifolds of negative sectional curvature are examples of Finsler manifolds of negative flag curvature.

To introduce our normalized entropy functional for Finsler manifolds, let $y$ be a point in such a space $Y$, $S_y(1,F) \subset T_yY$ its Finsler unit sphere, and $u \in S_y(1,F)$ a $F$-unit vector. Let $w_u \in S(1,g_u)$ be such that

$$F(y,w_u) = \max_{v\in S(1,g_u)} F(y,v).$$

Set

$$N(y) = \max_{u\in S_y(1,F)} \frac{F(y,w_u)^n}{\rho(u)}$$

and

$$N(F) = \max_{y\in Y} N(y).$$

Our normalized entropy functional is then

$$N(F)h(F)^n\text{Vol}(Y,F).$$

With this notation, our main result is

**Main Theorem.** *Let $(X,g_o)$ be a compact, $n$-dimensional, locally symmetric, Riemannian manifold of negative curvature ($n \geq 3$) and $(Y,F)$ a compact, reversible, Finsler manifold of negative flag curvature that is homotopy equivalent to $(X,g_o)$. Then*

*(i)* $N(g_o)h(g_o)^n\text{Vol}(X,g_o) \leq N(F)h(F)^n\text{Vol}(Y,F)$, *and*

*(ii)* $N(g_o)h(g_o)^n\mathrm{Vol}(X, g_o) = N(F)h(F)^n\mathrm{Vol}(Y, F)$ *holds if and only if* $(Y, F)$ *is Riemannian and homothetic to* $(X, g_o)$.

Notice that when $(Y, F)$ is Riemannian then $N(F) = N(g_o) = 1$ and we recover Theorem 1.

*Remark.* The conjecture in [4] is that this main theorem remains true with the normalized entropy functional $N(F)h(F)^n\mathrm{Vol}(Y, F)$ replaced by $h(F)^n\mathrm{Vol}(Y, F)$. This conjecture is not a simple consequence of our main theorem, however, since one may construct examples of Finsler manifolds satisfying the hypotheses of the theorem, but for which $N(F) > 1$. To see this, consider the example studied by D. Bao and Z. Shen [2] of a norm in $\mathbb{R}^2$ not derived from an inner product: for $(y_1, y_2) \in \mathbb{R}^2$ and $c \in \mathbb{R}^+$, let

$$F(y_1, y_2) = \sqrt{y_1^2 + y_2^2 + c\sqrt{y_1^4 + y_2^4}}.$$

Straightforward numerical methods show that for this example,

$$\max_{u \in S(1,F)} \frac{F(w_u)^n}{\rho(u)} > 1.$$

Now let $(M, g_o)$ be a compact hyperbolic surface and $p \in M$. Choose coordinates around $p$ so that the coefficients $g_{ij}$ of the metric tensor satisfy $g_{11}(p) = 1$, $g_{12}(p) = g_{21}(p) = 0$, and $g_{22}(p) = 1$. Let $\phi$ be a $C^\infty$ non-negative bump function with support in the coordinate patch $U$ such that $\phi$ and all its derivatives are uniformly small. Define a Finsler structure $F$ on $U$ by

$$F(x, y_1, y_2) = \sqrt{g_{11}(x)y_1^2 + 2g_{12}(x)y_1y_2 + g_{22}(x)y_2^2 + \phi(x)\sqrt{g_{11}(x)y_1^4 + g_{22}(x)y_2^4}}$$

where $x \in U$ and $(y_1, y_2) \in T_xM$. Combining this with the hyperbolic metric on $M \setminus U$ gives a smooth Finsler structure on $M$ which is a small perturbation of the hyperbolic metric, and so has negative flag curvature. At the point $p$, $F(p, y_1, y_2)$ is just the example mentioned above, so $N(F) \geq N(p) > 1$, as claimed.

We also remark that the first author has shown in [5] that if $F_t$ is a smooth deformation of a locally symmetric metric through Finsler metrics, then the functional $h(F_t)^n\mathrm{Vol}(X, F_t)$ has a critical point at $t = 0$, giving credence to the conjecture in [4], at least locally near the symmetric metric.

## 2. Proof of the Main Theorem

The method of proof follows that of Theorem 1: starting with a point $y$ in the universal cover $\tilde{Y}$, we push forward the Patterson-Sullivan measure $\mu_y$ from the boundary of $\tilde{Y}$ to the boundary of $\tilde{X}$, and use the barycenter map to arrive at a point $\tilde{\Phi}(y)$ in $\tilde{X}$. The main work consists in estimating the Jacobian of this map in terms of the volume growth entropies of $Y$ and $X$ and the Finsler structure on $Y$ (Proposition 4).

We begin with a homotopy equivalence $f : Y \to X$. The extension of the Cartan-Hadamard theorem to simply connected Finsler manifolds of negative flag curvature ([9] Proposition 3.1) implies that $\tilde{Y}$ is diffeomorphic to $\mathbb{R}^n$. There is an isomorphism $\tau : \pi_1(Y) \to \pi_1(X)$ and a lift of $f$ to a map $\tilde{f} : \tilde{Y} \to \tilde{X}$ satisfying $\tilde{f}(\gamma(y)) = \tau(\gamma)\tilde{f}(y)$ for all $y \in \tilde{Y}$ and $\gamma \in \pi_1(Y)$. Since $\tilde{Y}$ and $\tilde{X}$ are quasi-isometric to $\pi_1(Y)$ and $\pi_1(X)$

respectively, the map $\tilde{f}$ is a quasi-isometry. By work of D. Egloff ([8] §5) this induces a homeomorphism on the boundaries at infinity

$$\bar{f} : \partial\tilde{Y} \to \partial\tilde{X}$$

satisfying $\bar{f} \circ \gamma = \tau(\gamma) \circ \bar{f}$.

We now construct the Patterson-Sullivan measures $\mu_y$, one for each $y \in \tilde{Y}$. Fix an origin $o \in \tilde{Y}$, set $\Gamma = \pi_1(Y)$, and define for each $s > h(F)$ the Poincaré series $g_s(y) = \sum_{\gamma\in\Gamma} e^{-sd(y,\gamma y)}$, where $d$ is the distance function associated to the lifted Finsler metric on $\tilde{Y}$. By the definition of the volume growth entropy, $g_s(y)$ diverges for $s < h(F)$ and converges for $s > h(F)$; the standard weighting function argument (see, for example, [11]) allows us to assume without loss of generality that $g_s(y)$ diverges for $s = h(F)$. Letting $\delta_z$ denote the Dirac measure at $z \in \tilde{Y}$, one defines for $s > h(F)$ and $y \in \tilde{Y}$ the measures

$$\nu_y^s = \frac{\Sigma_{\gamma\in\Gamma} e^{-sd(y,\gamma o)}\delta_{\gamma o}}{g_s(o)}.$$

Using the triangle inequality one observes that for $s$ near $h(F)$, $\nu_y$ has total mass bounded independently of $s$, so for a fixed $y$, we may take a weak limit of these measures along a subsequence $s_i \to h(F)^+$ to obtain a measure $\nu_y$ on $\partial\tilde{Y}$. As pointed out in [15], for any other $z \in \tilde{Y}$ the same subsequence $s_i$ produces a limit $\nu_z$ absolutely continuous with respect to $\nu_y$. To express the Radon-Nikodym derivative, we recall the definition of the Busemann functions (which continue to exist and have the usual properties in the Finsler case, see [9]). For $\theta \in \partial\tilde{Y}$ a point at infinity, $\sigma(t)$ a geodesic ray starting at the origin $o$ and asymptotic to $\theta$, the Busemann function $B(y, \theta)$ is given by

$$B(y, \theta) = \lim_{t\to\infty} (d(y, \sigma(t)) - t).$$

Then the Radon-Nikodym derivative for the measures $\nu_y$ is

$$\frac{d\nu_y}{d\nu_o}(\theta) = e^{-h(F)B(y,\theta)}.$$

By their definition, the measures $\nu_y$ are equivariant under covering isometries $\gamma$ of $\tilde{Y}$: $\gamma_*\nu_y = \nu_{\gamma y}$. Letting $c(y)$ be the total mass of $\nu_y$, the Patterson-Sullivan measures $\mu_y$ are the probability measures $\frac{\nu_y}{c(y)}$. They are equivariant under isometries and satisfy

$$\frac{d\mu_y}{d\mu_o}(\theta) = \frac{c(o)}{c(y)} e^{-h(F)B(y,\theta)}.$$

Finally, using the Radon-Nikodym property one can show (as in [16]) that they have no atoms.

In [3], the authors introduce the barycenter map taking a measure $\lambda$ without atoms on the boundary of the universal cover $\tilde{X}$ of a negatively curved, Riemannian manifold to its "barycenter" $\mathrm{bar}(\lambda) \in \tilde{X}$, defined to be the unique point $x \in \tilde{X}$ at which the function

$$\mathcal{B}(x) = \int_{\partial\tilde{X}} B_o(x, \theta) d\lambda[\theta] \tag{1}$$

attains its minimum, where $B_o$ is the Busemann function on $\tilde{X}$. We get nonatomic measures on $\partial\tilde{X}$ by pushing-forward the Patterson-Sullivan measures $\mu_y$ using $\bar{f}$.

Applying the barycenter map to these push-forward measures gives us the *natural map* $\tilde{\Phi} : \tilde{Y} \to \tilde{X}$,

$$\tilde{\Phi}(y) = \mathrm{bar}(\bar{f}_*\mu_y).$$

By the equivariance properties of the Patterson-Sullivan measures and the boundary map $\bar{f}$, this descends to a map $\Phi : Y \to X$, which we also call the natural map. It is homotopic to $f$ since it induces the isomorphism $\tau$ between the fundamental groups of $Y$ and $X$.

Let Jac $\tilde{\Phi}$ denote the Jacobian of $\tilde{\Phi}$ computed with respect to the Finsler volume form $dF$ and the Riemannian volume form $dg_o$ on $\tilde{X}$. The following gives a pointwise estimate on this Jacobian in terms of the entropies of the two manifolds and the Finsler structure.

**Proposition 4.** *For $y \in \tilde{Y}$ and each nonzero $u \in T_y\tilde{Y}$,*

*(i)* $|Jac\ \tilde{\Phi}(y)| \leq \frac{h(F)^n}{h(g_o)^n}\frac{F(y,w_u)^n}{\rho(u)}$

*(ii) If* $|Jac\ \tilde{\Phi}(y)| = \frac{h(F)^n}{h(g_o)^n}\frac{F(y,w_u)^n}{\rho(u)}$, *then with respect to $g_u$ on $T_y\tilde{Y}$ and $g_o$ on $T_{\tilde{\Phi}(y)}\tilde{X}$, $D_y\tilde{\Phi}$ is an isometry composed with a homothety of ratio* $\left(\frac{h(F)}{h(g_o)}\right)F(y,w_u)$.

*Proof.* Fix $u$ nonzero in $T_y\tilde{Y}$. The proof follows that of ([4] Proposition 5.2), except that $(Y,F)$ is not Riemannian, so we are forced to use the inner product $g_u$ on $T_yY$. This complicates matters because the tangent fields to $F$-geodesics passing through $y$ are not $g_u$-unit vectors.

We begin by obtaining an expression involving the differential $D_y\tilde{\Phi}$ of the natural map $\tilde{\Phi}$. Because the barycenter is the critical point of $\mathcal{B}(x)$, $\tilde{\Phi}$ is defined implicitly by the vector valued equation

$$\int_{\partial\tilde{X}} (dB_o)_{(\tilde{\Phi}(y),\theta)}(\cdot)d(\bar{f}_*(\mu_y))[\theta] = 0,$$

or by changing variables

$$\int_{\partial\tilde{Y}} (dB_o)_{(\tilde{\Phi}(y),\bar{f}(\alpha))}(\cdot)e^{-h(F)B(y,\alpha)}d\mu_o[\alpha] = 0. \tag{2}$$

Since the Busemann functions $B_o$ and $B$ are $C^1$ in their first variable (see [9] for the Finsler case), following [4] we can differentiate equation (2) via the implicit function theorem and conclude that the natural map $\tilde{\Phi}$ is $C^1$, and for $y \in \tilde{Y}$, $v \in T_y\tilde{Y}$, and $w \in T_{\tilde{\Phi}(y)}\tilde{X}$, the differential $D_y\tilde{\Phi}$ satisfies

$$\begin{aligned}\int_{\partial\tilde{Y}} DdB_{o(\tilde{\Phi}(y),\bar{f}(\alpha))}(D_y\tilde{\Phi}(v),w)d\mu_y[\alpha] \\ = h(F)\int_{\partial\tilde{Y}} dB_{o(\tilde{\Phi}(y),\bar{f}(\alpha))}(w)dB_{(y,\alpha)}(v)d\mu_y[\alpha].\end{aligned} \tag{3}$$

As in [4], we introduce endomorphisms $K$ and $H$ on $T_{\tilde{\Phi}(y)}\tilde{X}$, expressed as bilinear forms with respect to $g_o$ as follows.

$$g_o(K(w), w') = \int_{\partial \tilde{X}} DdB_{o(\tilde{\Phi}(y),\theta)}(w, w')d(\bar{f}_*\mu_y)[\theta]$$
$$g_o(H(w), w) = \int_{\partial \tilde{X}} dB_{o(\tilde{\Phi}(y),\theta)}^{\;2}(w)d(\bar{f}_*\mu_y)[\theta].$$

With this notation, the left hand side of equation (3) becomes $g_o(K \circ D_y\tilde{\Phi}(v), w)$, and applying the Cauchy-Schwartz inequality to equation (3) yields our desired expression involving $D_y\tilde{\Phi}$:

$$\left|g_o\left(K \circ D_y\tilde{\Phi}(v), w\right)\right| \leq h(F)g_o\left(H(w), w\right)^{1/2}\left(\int_{\partial \tilde{Y}} dB^2_{(y,\alpha)}(v)d\mu_y[\alpha]\right)^{1/2}, \tag{4}$$

for $w \in T_{\tilde{\Phi}(y)}\tilde{X}$ and $v \in T_y\tilde{Y}$.

Our next task is to prove the following inequality, which relates $H$ and $K$ to the Jacobian of $\tilde{\Phi}$:

$$|\text{Jac } \tilde{\Phi}(y)| \leq \frac{h(F)^n}{n^{n/2}} \frac{(\det H)^{1/2}}{\det K} \frac{F(y, w_u)^n}{\rho(u)}, \tag{5}$$

where the determinants are taken with respect to an orthonormal basis for $T_{\tilde{\Phi}(y)}\tilde{X}$. Note that we can assume that $D_y\tilde{\Phi}$ has maximal rank, since if not then Jac $\tilde{\Phi}(y)$=0 and the inequality is trivial; hence we assume without loss of generality that $D_y\tilde{\Phi}$ is invertible. Furthermore, $K$ is invertible because the function $\mathcal{B}(x)$ from equation (1) is strictly convex [4] and has the bilinear form $g_o(K(\cdot), \cdot)$ as its Hessian.

Fix an orthonormal basis $\{e_i\}$ for $T_{\tilde{\Phi}(y)}\tilde{X}$ such that the representation of the endomorphism $H$ with respect to $\{e_i\}$ is a diagonal matrix, and let $v_i' = (K \circ D_y\tilde{\Phi})^{-1}(e_i)$. Apply the Gram-Schmidt procedure to this basis to construct a new basis $\{v_i\}$ for $T_y\tilde{Y}$ that is orthonormal with respect to the inner product $g_u$. Then with respect to $\{v_i\}$ and $\{e_i\}$, the matrix of $K \circ D_y\tilde{\Phi}$ is triangular.

Now Jac $\tilde{\Phi}(y)$ is just the ratio of the determinant of the matrix of $D_y\tilde{\Phi}$ (with respect to the bases $\{e_i\}$ and $\{v_i\}$) to the $F$-volume of the parallelepiped spanned by the $\{v_i\}$. Recall that the Finsler volume form is given by

$$dF(y) = \frac{C_n}{\text{Vol}_g(B_y(1, F))}\, dg$$

for any background Riemannian metric $g$ on $\tilde{Y}$. We choose a background metric that agrees at $y$ with $g_u$, so that the parallelepiped spanned by $\{v_i\}$ has volume 1 with respect to $g_u$. Then the Finsler volume of the parallelepiped spanned by the $\{v_i\}$ reduces to $\frac{C_n}{\text{Vol}_{g_u}(B_y(1,F))} = \rho(u)$, so we have $|\text{Jac } \tilde{\Phi}(y)| = \frac{|\det(D_y\tilde{\Phi})|}{\rho(u)}$.

Since $K \circ D_y \tilde{\Phi}$ is triangular and $H$ is diagonal, equation (4) yields

(6)
$$\begin{aligned} |\text{Jac } \tilde{\Phi}(y)| \; \rho(u) \det K &= |\det(K \circ D_y \tilde{\Phi})| \\ &= \prod_{i=1}^{n} |g_o(K \circ D_y \tilde{\Phi}(v_i), e_i)| \\ &\le h(F)^n \prod_{i=1}^{n} g_o\left(H(e_i), e_i\right)^{1/2} \prod_{i=1}^{n} \left( \int\limits_{\partial \tilde{Y}} dB^2_{(y,\alpha)}(v_i) d\mu_y[\alpha] \right)^{1/2} \\ &\le h(F)^n (\det H)^{1/2} \left( \frac{1}{n} \sum_{i=1}^{n} \int\limits_{\partial \tilde{Y}} dB^2_{(y,\alpha)}(v_i) d\mu_y[\alpha] \right)^{n/2} . \end{aligned}$$

The $\{v_i\}$ are orthonormal with respect to $g_u$, so $\sum_{i=1}^{n} dB^2_{(y,\alpha)}(v_i) = \|dB_{(\tilde{\Phi}(y),\alpha)}\|^2_{g_u}$. If $\tilde{Y}$ were Riemannian, this norm would be with respect to the Riemannian metric on $\tilde{Y}$ and the Busemann function would be constructed using the geodesics of that metric, so the value would be 1. In our case, the Busemann function is constructed using $F$-geodesics, and the $g_u$ norm of its gradient is not in general 1, so we are forced to estimate. For $v \in T_y\tilde{Y} \backslash \{0\}$, let $\hat{v} = \frac{v}{F(y,v)}$ be the $F$-unit vector in the direction of $v$. Then

$$\|dB_{(y,\alpha)}\|_{g_u} = \max_{v \in S(1,g_u)} dB_{(y,\alpha)} v = \max_{v \in S(1,g_u)} F(y,v) dB_{(y,\alpha)} \hat{v},$$

so

(7)
$$\begin{aligned} \sum_{i=1}^{n} \int\limits_{\partial \tilde{Y}} dB^2_{(y,\alpha)}(v_i) d\mu_y[\alpha] &= \int\limits_{\partial \tilde{Y}} \|dB_{(y,\alpha)}\|^2_{g_u} d\mu_y[\alpha] \\ &\le \left( \max_{v \in S(1,g_u)} F(y,v) \; \max_{\alpha \in \partial \tilde{Y}} dB_{(y,\alpha)} \hat{v} \right)^2 \\ &= \max_{v \in S(1,g_u)} F(y,v)^2 = F(y,w_u)^2, \end{aligned}$$

since $\mu_y$ is a probability measure, and the maximum of $dB_{(y,\alpha)}\hat{v}$ over $\alpha$ is attained when $\alpha$ is the endpoint of the $F$-geodesic ray tangent to $\hat{v}$, and for that $\alpha$, $dB_{(y,\alpha)}\hat{v} = 1$. Substituting into equation (6), we arrive at

$$|\text{Jac } \tilde{\Phi}(y)| \; \rho(u) \det K \le \frac{h(F)^n}{n^{n/2}} \; (\det H)^{1/2} F(y,w_u)^n,$$

verifying equation (5).

In order to finish the proof of the proposition, we appeal to the following linear algebraic lemma from [3], which we note requires that $n \ge 3$.

**Lemma 5.** *For $H$ and $K$ the $n \times n$ matrices defined above,*

*(i)* $\frac{(\det H)^{1/2}}{\det K} \le \left( \frac{\sqrt{n}}{h(g_o)} \right)^n$, *and*

*(ii) if* $\frac{(\det H)^{1/2}}{\det K} = \left( \frac{\sqrt{n}}{h(g_o)} \right)^n$, *then* $H = \frac{1}{n} I$, *and* $K = \frac{h(g_o)}{n} I$.

Applying the inequality of Lemma 5 to equation (5) yields the inequality of Proposition 4. For the statement of equality in Proposition 4, we have

$$\begin{aligned}\frac{h(F)^n}{h(g_o)^n}\frac{F(y,w_u)^n}{\rho(u)} &= |\mathrm{Jac}\ \tilde\Phi(y)| \\ &\le \frac{h(F)^n}{n^{n/2}}\ \frac{(\det H)^{1/2}}{\det K}\frac{F(y,w_u)^n}{\rho(u)} \\ &\le \frac{h(F)^n}{h(g_o)^n}\frac{F(y,w_u)^n}{\rho(u)},\end{aligned}$$

putting us in the equality case of Lemma 5. Then equation (4) becomes

$$g_o(D_y\tilde\Phi(v),w) \le n^{1/2}\left(\frac{h(F)}{h(g_o)}\right)\|w\|_{g_o}\left(\int_{\partial\tilde Y} dB^2_{(y,\alpha)}(v)d\mu_y[\alpha]\right)^{1/2}.$$

By taking the supremum over all $w \in T_{\tilde\Phi(y)}\tilde X$ such that $\|w\|_{g_o}=1$, this yields

$$\|D_y\tilde\Phi(v)\|_{g_o} \le n^{1/2}\left(\frac{h(F)}{h(g_o)}\right)\left(\int_{\partial\tilde Y} dB^2_{(y,\alpha)}(v)d\mu_y[\alpha]\right)^{1/2},$$

for all $v \in T_y\tilde Y$.

Let $L=(D_y\tilde\Phi)^*(D_y\tilde\Phi)$ and $\{v_i\}$ be an orthonormal basis of $T_y\tilde Y$ for $g_u$. Then using equation (7),

$$\begin{aligned}\mathrm{trace}(L) = \sum_{i=1}^n g_u(Lv_i,v_i) &= \sum_{i=1}^n g_o(D_y\tilde\Phi(v_i),D_y\tilde\Phi(v_i)) \\ &= \sum_{i=1}^n \|D_y\tilde\Phi(v_i)\|^2_{g_o} \\ &\le n\left(\frac{h(F)}{h(g_o)}\right)^2\left(\sum_{i=1}^n\int_{\partial\tilde Y} dB^2_{(y,\alpha)}(v_i)d\mu_y[\alpha]\right) \\ &\le n\left(\frac{h(F)}{h(g_o)}\right)^2 F(y,w_u)^2.\end{aligned}$$

Since we are in the equality case,

$$\begin{aligned}\left(\frac{h(F)}{h(g_o)}\right)^{2n}\frac{F(y,w_u)^{2n}}{\rho(u)^2} &= |\mathrm{Jac}\ \tilde\Phi(y)|^2 \\ &= \frac{\det L}{\rho(u)^2} \\ &\le \frac{1}{\rho(u)^2}\left(\frac{\mathrm{trace}(L)}{n}\right)^n \le \left(\frac{h(F)}{h(g_o)}\right)^{2n}\frac{F(y,w_u)^{2n}}{\rho(u)^2},\end{aligned}$$

so $\det L = \left(\frac{\mathrm{trace} L}{n}\right)^n$, which implies $L = \frac{h(F)^2}{h(g_o)^2}F(y,w_u)^2 I$. Hence $D_y\tilde\Phi$ is an isometry between $(T_y\tilde Y, g_u)$ and $(T_{\tilde\Phi(y)}\tilde X, g_o)$, composed with a homothety of ratio $\frac{h(F)}{h(g_o)}F(y,w_u)$, as desired. □

*Proof of the Main Theorem.* For every $y \in Y$ and $u \in T_yY$, Proposition 4 implies that

$$|\text{Jac } \Phi(y)| \leq \frac{h(F)^n}{h(g_o)^n} \frac{F(y, w_u)^n}{\rho(u)} \leq \frac{h(F)^n}{h(g_o)^n} N(F).$$

Let $N_\Phi(x)$ denote the number of points in $(Y, F)$ that map to $x$ under $\Phi$. Note that $\Phi$ is a surjection since it is homotopic to the homotopy equivalence $f$, thus we can apply the co-area formula to obtain

$$\text{Vol}(X, g_o) \leq \int_X N_\Phi(x) dg_o = \int_Y |\text{Jac } \Phi(y)| dF \leq \frac{h(F)^n}{h(g_o)^n} N(F) \text{Vol}(Y, F).$$

Since $N(g_o) = 1$, this yields the required inequality.

In the equality case, we have $\text{Vol}(X, g_o) = \frac{h(F)^n}{h(g_o)^n} N(F) \text{Vol}(Y, F)$. Since $h(g_o)^n \text{Vol}(X, g_o)$ is unchanged if $g_o$ is scaled by a constant, we may assume without loss of generality that $\text{Vol}(Y, F) = \text{Vol}(X, g_o)$ and hence $\frac{h(F)^n}{h(g_o)^n} N(F) = 1$. So for every $y \in Y$ and nonzero $u \in T_yY$, we have

$$|\text{Jac } \Phi(y)| = \frac{h(F)^n}{h(g_o)^n} \frac{F(y, w_u)^n}{\rho(u)} = \frac{h(F)^n}{h(g_o)^n} N(F) = 1.$$

This puts us in the equality case of Proposition 4, so for all nonzero $u \in T_yY$, $D_y\Phi : T_yY \to T_{\Phi(y)}X$ is an isometry composed with a homothety of ratio $\frac{h(F)}{h(g_o)} F(y, w_u)$ with respect to $g_u$ and $g_o$. This implies that for every $u \in T_yY$, $D_y\Phi$ maps $S(1, g_u)$ to $S(\frac{h(F)}{h(g_o)} F(y, w_u), g_o)$. Now for each $u$ in the Finsler unit ball, $S_y(1, F)$ is tangent to $S(1, g_u)$ at $u$ (see [12] §3). Thus by linearity of $D_y\Phi$, for each such $u$, $D_y\Phi(S_y(1, F))$ is tangent to $S(\frac{h(F)}{h(g_o)} F(y, w_u), g_o)$ at $D_y\Phi(u)$. An exercise in calculus gives us

**Lemma 6.** *If $f$ is a $C^1$ function on $\mathbb{R}$ whose tangent line at $(x, f(x))$ is orthogonal to the vector $(x, f(x))$ for every $x \in$ Domain$(f)$, then $f$ is an arc of a circle.*

Since $D_y\Phi(S_y(1, F))$ is at each point $D_y\Phi(u)$ tangent to the sphere $S(\frac{h(F)}{h(g_o)} F(y, w_u), g_o)$, the lemma tells us that the projection onto any two dimensional subspace traces out a circle, i.e. $D_y\Phi(S_y(1, F))$ must itself be a sphere. But from the tangency, this implies that $\frac{h(F)}{h(g_o)} F(y, w_u) = \rho(u)^{-1/n}$ is constant in $u$. By Proposition 2, $F(y, \cdot)$ is Euclidean and $\rho(u) = 1$ for each $u$. Since this is true for each $y \in \tilde{Y}$, $F$ is Riemannian. Moreover, $D_y\Phi$ is a homothety of ratio $\frac{h(F)}{h(g_o)} F(y, w_u) = \rho(u)^{-1/n} = 1$, i.e. an isometry, concluding the proof of the theorem. ☐

## References


1. A. Abate and G. Patrizio, *Finsler metrics–a global approach*, Lecture Notes in Mathematics, vol. 1591, Springer, 1994.
2. D. Bao and Z. Shen, *On the volume of unit tangent spheres in a Finsler manifold*, Results in Mathematics **26** (1994), 1–17.
3. G. Besson, G. Courtois, and S. Gallot, *Entropies et rigiditiés des espaces localement symétrique de courbure strictement négative.*, G.A.F.A. **5** (1995), no. 5, 731–799.
4. ———, *Minimal entropy and Mostow's rigidity theorems*, Ergod. Th. & Dynam. Sys. **16** (1996), 623–649.
5. J. Boland, *The geometry and dynamics of contact Anosov flows*, Ph.D. thesis, University of Michigan, 1998.
6. F. Brickell, *A new proof of Deicke's theorem on homogeneous functions*, Proc. Amer. Math. Soc. **16** (1965), 190–191.

7. A. Deicke, *Über die Finsler-räume mit $A_i = 0$*, Arch. Math. **4** (1953), 45–51.
8. D. Egloff, *On the dynamics of uniform Finsler manifolds of negative flag curvature*, Ann. Global Anal. Geom. **15** (1997), 101–116.
9. ______, *Uniform Finsler Hadamard manifolds*, Ann. Inst. H. Poincaré Phys. Théor. **66** (1997), no. 3, 323–357.
10. A. Katok, *Entropy and closed geodesics*, Ergod. Th. & Dynam. Sys. **2** (1982), 339–367.
11. P. Nicholls, *The Ergodic Theory of Discrete Groups*, Cambridge U. Press: Cambridge, 1989.
12. H. Rund, *The Differential Geometry of Finsler spaces*, Springer, 1959.
13. Z. Shen, *Volume comparison and its applications in Riemann-Finsler geometry*, Adv. Math. **128** (1997), 306–328.
14. ______, *Differential geometry of spray and Finsler spaces*, manuscript, 275 pages, 1999.
15. D. Sullivan, *The density at infinity of a discrete group of hyperbolic motions*, Publ. Math. I.H.E.S. **50** (1979), 171–202.
16. C. Yue, *The ergodic theory of discrete isometry groups on manifolds of variable negative curvature*, Trans. AMS **348** (1996), no. 12, 4965–5005.

Jeff Boland, Mathematics and Statistics, McMaster University, Hamilton, ON L8S 4K1 Canada
*E-mail address*: bolandj@icarus.math.mcmaster.ca

Florence Newberger, Dept of Mathematics, Pennsylvania State University, State College, PA
*E-mail address*: fan@math.psu.edu