DUAL SPHERES HAVE THE SAME GIRTH

J.C. ÁLVAREZ PAIVA

Abstract. Symplectic geometry is used to study the inner geometry of unit spheres in normed spaces. In particular, it is proved that the girth of a normed space — the infimum of the the lengths of all curves on the unit sphere that join a pair of antipodes — equals the girth of its dual.

Quidquid latet apparebit.
Mozart's Requiem.

1. Introduction

The starting point of this paper is a theorem of Schäffer (see [6]) stating that the length of the unit circle of a two-dimensional normed space equals the length of the unit circle of its dual. Here, as in the rest of the paper, the length of a continuous curve $\gamma : [a, b] \to V$ in a normed space $(V, \|\cdot\|)$ is defined as

$$\sup\{\sum_{i=0}^{k-1} \|\gamma(t_{i+1}) - \gamma(t_i)\| : a = t_0 < \cdots < t_k = b \text{ is a partition of } [a, b]\},$$

whenever this quantity exists.

In attempting to extend this result to higher dimensions, one must consider that the length of the unit circle is at once its surface area, the length of its shortest closed geodesic, twice the infimum of the lengths of all curves that join a pair of antipodes, and twice its inner diameter. Of all these interpretations, Schäffer singled out the third for further study. More precisely, he defined the *girth* of a normed space as the infimum of the lengths of all curves on the unit sphere that join a pair of antipodes and conjectured in his book ([7]) that the girth of a normed space equals the girth of its dual. Admitting that he had little evidence to go on, he nevertheless showed that the the conjecture had only to be verified in the finite-dimensional setting: if a normed space is such that the conjecture holds for every one of its finite-dimensional subspaces, then its girth is equal to the girth of its dual.

Schäffer also proved in [7], pp. 91, that the girth, as an invariant of isometry classes of normed spaces, is continuous with respect to the topology given by the Banach-Mazur distance. This property reduces the problem to verifying the conjecture in the case of finite-dimensional normed spaces whose spheres and their duals are smooth. Following the standard terminology in Finsler geometry, we shall call these spaces *Minkowski spaces.* In this paper, we settle Schäffer's conjecture by proving the following result:

1991 *Mathematics Subject Classification.* 46B20.

Key words and phrases. Convex geometry, geometry of normed spaces, symplectic geometry, integral geometry.

This work was partially funded by a *crédit aux chercheurs* from the FNRS.

Theorem 1.1. *The girth of a Minkowski space equals the girth of its dual.*

Along with this theorem, we'll see that the area and the length spectrum — the set of lengths of all closed geodesics — of the unit sphere of a Minkowski space are respectively equal to the area and the length spectrum of the unit sphere of its dual. In particular, the first three possible extensions of Schäffer's two-dimensional result are all true. Two important remarks are in order. The first is that the fourth possible extension (i.e., that the inner diameter of the unit sphere of a normed space and that of its dual are equal) is false. The interested reader will find a counterexample in page 110 of Schäffer's book. The second is that the statement for the areas was first proved by Holmes and Thompson in [5]. It was their work that directly inspired the present investigations.

In [5], Holmes and Thompson discovered that something more general than convex duality was at play. Namely, they showed that if $\|\cdot\|_1$ and $\|\cdot\|_2$ are two norms on $\mathbb{R}^n$ while $\|\cdot\|_1^*$ and $\|\cdot\|_2^*$ denote their dual norms on $\mathbb{R}^{n*}$, then the area of the unit sphere of $(\mathbb{R}^n, \|\cdot\|_1)$ considered as a submanifold of $(\mathbb{R}^n, \|\cdot\|_2)$ equals the area of the unit sphere of $(\mathbb{R}^{n*}, \|\cdot\|_2^*)$ considered as a submanifold of $(\mathbb{R}^{n*}, \|\cdot\|_1^*)$. We will see that all three higher-dimensional extensions of Schäffer's theorem can be generalized in this way.

The proof of the results in this paper is based on the following simple construction: let V be a vector space and let $S \subset V$ and $\Sigma \subset V^*$ be smooth, quadratically convex hypersurfaces. Let S_Σ and Σ_S respectively denote the boundaries of the sets

$$\{(q, P_{|[T_qS]}) : q \in S, P \in \Sigma\} \subset T^*S \text{ and } \{(P, q_{|[T_P\Sigma]}) : P \in \Sigma, q \in S\} \subset T^*\Sigma,$$

where the notation $[T_qS]$ and $[T_P\Sigma]$ stresses that the tangent spaces of S at q and of Σ at P are considered as vector subspaces.

The map ϕ that assigns to each covector $(q,p) \in S_\Sigma$ the unique covector $(P,Q) \in \Sigma_S$ such that $q_{|[T_P\Sigma]} = Q$ and $P_{|[T_qS]} = p$ is a diffeomorphism between S_Σ and Σ_S. Moreover, it has the following important properties:

Lemma 1.1. *If α denotes the canonical 1-form on S_Σ and β denotes the canonical 1-form on Σ_S, then $\phi^*\beta = -\alpha + df$. Here $f : S_\Sigma \to \mathbb{R}$ is the function that sends a unit covector (q,p) to the number $P \cdot q$, where $\phi(q,p) = (P,Q)$.*

Lemma 1.2. *The map ϕ extends continuously to a homeomorphism between the sets*

$$\{(q, P_{|[T_qS]}) : q \in S, P \in \Sigma\} \text{ and } \{(P, q_{|[T_P\Sigma]}) : P \in \Sigma, q \in S\}.$$

Moreover, the restriction of this homeomorphism to the interiors of both sets is an anti-symplectic diffeomorphism.

In section 4 we will use lemma 1.2 to prove that the area of convex hypersurface multiplied by the area of the Euclidean unit ball of dimension $n-1$ equals the measure of the set of lines that intersects the hypersurface. A result previously obtained by El-Ekhtiar (see [4]) using other techniques.

The plan of the paper is as follows: in section 2, we introduce some basic concepts in Finsler geometry from the symplectic viewpoint. Section 3 contains the proofs of theorem 1.1 and its generalizations assuming the previous lemmas. In section 4, we uncover the symplectic origins of El-Ekhtiar's theorem. Section 5 contains the nearly trivial proof of lemmas 1.1 and 1.2.

Acknowledgments . The author warmly thanks Erwin Lutwak and Tony Thompson for introducing him to the beautiful subject of Minkowski geometry. Conversations with R. Schneider and D. Hug greatly helped with the final touches of this work.

2. Finsler geometry from the symplectic viewpoint

Coarsely speaking, a Finsler manifold is a manifold together with the choice of a norm on each tangent space. The precise definition requires us to restrict the class of norms to those where the unit sphere is smooth and *quadratically convex:* the principal curvatures are positive for some (and therefore any) auxiliary Euclidean structure. The intrinsic description of these norms is as follows:

Let V be a vector space and let $\varphi : V \to [0, \infty)$ be a norm which is smooth outside the origin. If v is a nonzero vector, we define the bilinear form $g_\varphi(v)$ evaluated at a pair of vectors w_1 and w_2 by taking a smooth vector valued function $\alpha(s,t)$ such that $\alpha(0,0) = v$, $\frac{\partial\alpha}{\partial s}(0,0) = w_1$, $\frac{\partial\alpha}{\partial t}(0,0) = w_2$, and setting

$$g_\varphi(v)(w_1, w_2) := \frac{1}{2}\frac{\partial^2(\varphi\circ\alpha)^2}{\partial s\partial t}(0,0).$$

Definition 2.1. A smooth norm $\varphi : V \to [0,\infty)$ is said to be a *Minkowski norm* if for every nonzero vector v, the bilinear form $g_\varphi(v)$ is positive definite. A finite-dimensional vector space provided with a Minkowski norm will be called a *Minkowski space*.

Definition 2.2. Let M be a smooth manifold and let $TM \setminus 0$ denote its tangent bundle with the zero section deleted. A *Finsler metric* on M is a smooth function

$$\varphi : TM \setminus 0 \longrightarrow \mathbb{R}$$

such that for each point $m \in M$ the restriction of φ to T_mM is a Minkowski norm.

A Finsler metric φ on a manifold M allows us to define the length of a smooth curve $\gamma : [a,b] \to M$ by the equation

$$\text{length of } \gamma := \int_a^b \varphi(\dot\gamma(t))dt.$$

Using this, we can define the distance between two points x and y in M as the infimum of the lengths of all smooth curves joining x and y.

While Finsler manifolds are an interesting class of metrics spaces and the methods of metric geometry can be applied to their study, the techniques that interest us here come from their relation to Hamiltonian systems. We refer the reader to the books by Abraham and Marsden ([1]), and Arnold and Givental ([3]) for the details of the descriptions that follow.

If (V,φ) is a normed space, then the dual vector space V^* inherits a natural norm defined by the equation

$$\varphi^*(P) := \sup\{|P\cdot v| : \varphi(v) \le 1\}.$$

On Minkowski spaces, a related construction is the *Legendre transform* which assigns to a nonzero vector $v \in V$ the covector defined by $g_\varphi(v)(v,\cdot)$. Note that if v belongs to the unit sphere $S \subset V$, then the Legendre transform of v is the unique covector P such that $P\cdot w = 1$ for all w belonging to the hyperplane tangent to S at the point v. Thus the image of S under the Legendre transform is the unit sphere in (V^*,φ^*).

Let (M, φ) be a Finsler manifold and for each point $m \in M$ let φ_m denote the Minkowski norm on $T_m M$. If $(T^*_m M, \varphi^*_m)$ is the dual of the normed space $(T_m M, \varphi_m)$, then the function

$$H : T^*M \longrightarrow \mathbb{R}$$

defined by $H(p_m) := \varphi^*_m(p_m)$ is a Hamiltonian whose energy surfaces are fiberwise convex. Applying the Legendre transform on each tangent space of M allows us to identify $TM \setminus 0$ and $T^*M \setminus 0$, and to pass from φ to H. This identification enables us to use the geometric structures present in the cotangent bundle: the canonical 1-form and the symplectic form.

Definition 2.3. The canonical 1-form α on the cotangent bundle of a smooth manifold M is the differential form whose value at a vector $v_{p_q} \in T_{p_q}T^*M$ is defined as follows: let $\bar{\gamma}$ be a smooth curve on T^*M such that its velocity at time zero equals v_{p_q} and let γ be its projection to M. We set $\alpha(v_{p_q}) := p_q \cdot \dot{\gamma}(0)$. The *symplectic 2-form* is defined as $\omega := -d\alpha$.

In this paper, the main object of study will be the *unit co-sphere bundle* $S^*_H M := H^{-1}(1)$ together with the restriction of the canonical 1-form α to $S^*_H M$, which we'll denote by α_H. We remark that α_H is a *contact form* (i.e., the top-order form $\alpha_H \wedge (d\alpha_H)^{n-1}$ never vanishes). Using α_H we can describe the geodesic flow, measure the lengths of smooth curves, and define the volume of the Finsler manifold.

Definition 2.4. The *Reeb vector field* X_H on $S^*_H M$ is defined by the equations

$$\alpha_H(X_H) = 1, \quad d\alpha_H(X_H, \cdot) = 0.$$

Proposition 2.1. *The projection to M of the integral curves of the Reeb vector field are geodesics parametrized with unit speed. Conversely, if γ is a geodesic on M parametrized with unit speed, then the Legendre transform maps the velocity curve $\dot{\gamma}$ to an integral curve of the Reeb vector field.*

Proposition 2.2. *If γ is any smooth curve on a Finsler manifold M that is parametrized with unit speed and $\bar{\gamma}$ is the image of $\dot{\gamma}$ under the Legendre transform, then*

$$\textit{length of } \gamma = \int_{\bar{\gamma}} \alpha_H. \tag{1}$$

We warn the reader that in the rest of the paper we use the term *geodesic* to denote both the curve on the Finsler manifold and its lift to the unit co-sphere bundle.

Definition 2.5. The *Holmes-Thompson volume* of an n-dimensional Finsler manifold M with Hamiltonian H equals the integral of $|\alpha_H \wedge (d\alpha_H)^{n-1}|$ over $S^*_H M$ divided by the volume of the Euclidean n-dimensional unit ball.

While this last definition of the Holmes-Thompson volume is not in its original form, it is not hard to see that it is equivalent to that given by Holmes and Thompson in [5] and [8] when the Finsler manifold is a submanifold of a Minkowski space.

Using the canonical 1-form, we can define a notion of equivalence between Finsler manifolds which is weaker than that of isometry.

Definition 2.6. Let M and N be two Finsler manifolds with unit co-sphere bundles S^*M and S^*N and canonical 1-forms α_M and α_N. The Finsler manifolds M and N will be said to be *alf-equivalent* if there exists a diffeomorphism

$$\phi : S^*M \longrightarrow S^*N$$

and a function $f : S^*M \to \mathbb{R}$ such that $\phi^*\alpha_N = \alpha_M + df$.

Clearly, two isometric Finsler manifolds are alf-equivalent (the strange name is a contraction of α and f) while the converse is not true. Nevertheless, alf-equivalent Finsler manifolds have similar geometric invariants:

Proposition 2.3. *If M and N are two alf-equivalent Finsler manifolds, then their volumes and their length spectra are equal.*

Proof. Since the geodesic distribution of a Finsler manifold is just the kernel of the restriction of the symplectic form to the unit co-sphere bundle, we have that the map ϕ defining the alf-equivalence takes geodesics to geodesics. The definition of length and volume in terms of the canonical 1-form implies that the lengths of closed geodesics and the volumes of both Finsler manifolds will be the same. □

In the next section, we will also consider the *unit co-disc bundle* of a Finsler manifold M with Hamiltonian H. This set is defined as the set of covectors $p_q \in T^*M$ such that $H(p_q) < 1$. It follows from Stokes theorem that the Holmes-Thompson volume of a Finsler manifold, as defined above, equals the symplectic volume of its unit co-disc bundle divided by the volume of the Euclidean n-dimensional unit ball.

3. Proof of theorem 1.1 and its generalizations

We start by establishing the relation between the concepts introduced in the previous section and the geometric construction leading to lemmas 1.1 and 1.2 in the introduction.

Proposition 3.1. *Let M be a smooth submanifold in the Minkowski space $(V, \|\cdot\|)$. If $\Sigma \subset V^*$ is the unit sphere of the dual norm, then the unit co-disc bundle and the unit co-sphere bundle of M are equal, respectively, to the interior and the boundary of the set $\{(q, P_{|[T_qM]}) : q \in M, P \in \Sigma\} \subset T^*M$.*

The proof of this proposition is an immediate consequence of the following well-known fact:

Lemma 3.1. *Let V be a finite-dimensional vector space and let $B \subset V$ be a convex body containing the origin in its interior. If $W \subset V$ is a proper vector subspace, then the polar body of $W \cap B$ in W^* is the image of the boundary of B under the projection $P \mapsto P_{|W}$ from V^* to W^*.*

Theorem 3.1. *Let $(\mathbb{R}^n, \|\cdot\|_1)$ and $(\mathbb{R}^n, \|\cdot\|_2)$ be two Minkowski spaces and let S_1 and S_2 denote their unit spheres. The unit co-disc bundle of the Finsler metric on S_1 induced by its embedding into $(\mathbb{R}^n, \|\cdot\|_2)$ is symplectomorphic to the unit co-disc bundle of the Finsler metric on S_2^* induced by its embedding into $(\mathbb{R}^n, \|\cdot\|_1^*)$.*

Proof. Applying lemma 3.1 and lemma 1.2 with $S := S_1$ and $\Sigma := S_2^*$, we have that there exists an anti-symplectic diffeomorphism Φ between the unit co-disc bundle of the Finsler metric on S_1 induced by its embedding into $(\mathbb{R}^n, \|\cdot\|_2)$ and the unit co-disc bundle of the Finsler metric on S_2^* induced by its embedding into $(\mathbb{R}^n, \|\cdot\|_1^*)$.

Composing Φ with the map that assings to each covector p_q in T^*S_1 its opposite, $-p_q$, we obtain a symplectomorphism between the unit co-disc bundles. □

Using the symplectic definition of the Holmes-Thompson volume given at the end of last section we recover the following result:

Corollary 3.1 (Holmes and Thompson, [5]). *The unit sphere of a Minkowski space and its dual have the same volume.*

Theorem 3.2. *Let $(\mathbb{R}^n, \|\cdot\|_1)$ and $(\mathbb{R}^n, \|\cdot\|_2)$ be two Minkowski spaces and let S_1 and S_2 denote their unit spheres. The Finsler metric on S_1 induced by its embedding into $(\mathbb{R}^n, \|\cdot\|_2)$ and the Finsler metric on S_2^* induced by its embedding into $(\mathbb{R}^n, \|\cdot\|_1^*)$ are alf-equivalent. Moreover, the diffeomorphism defining the alf-equivalence can be taken such that it takes centrally symmetric closed geodesics to centrally symmetric closed geodesics.*

Proof. Apply lemma 3.1 and lemma 1.1 with $S := S_1$ and $\Sigma := S_2^*$ to obtain a map ϕ satisfying $\phi^*\alpha_2 = -\alpha_1 + df$, where α_i denotes the canonical 1-form on the unit co-sphere bundle $S^*S_i, i = 1, 2$. Composing ϕ with the map σ that assigns to each covector p_q in T^*S_1 its opposite,$-p_q$, we obtain the desired alf-equivalence.

We still have to check that $\phi \circ \sigma$ takes centrally symmetric geodesics to centrally symmetric geodesics. For this purpose, let $a_i : S_i \to S_i, i = 1, 2$, denote the antipodal map and notice that $\phi \circ \sigma \circ a_1^* = a_2^* \circ \phi \circ \sigma$. Since the lifts to the unit co-sphere bundle of centrally geometric geodesics on S_i are precisely those integral curves of the Reeb vector field that are invariant under a_i^*, this is enough to see that centrally symmetric geodesics are taken to centrally symmetric geodesics. □

We are now ready for the proof of girth conjecture:

Proof. [Proof of theorem 1.1] Let V be a Minkowski space with unit sphere S and let Σ denote the unit sphere of its dual. By the previous theorem, S and Σ with their inherited Finsler metrics are alf-equivalent. Since the map defining the alf-equivalence takes closed centrally symmetric geodesics to closed centrally symmetrics geodesics of the same length, the length of the shortest closed centrally symmetric geodesic on S equals the length of the shortest closed centrally symmetric geodesic on Σ. This is tantamount to equality of the girths of V and V^*. □

4. Integral geometry

In this section we uncover the symplectic origins of a theorem of El-Ekhtiar (see [4]) on the integral geometry of finite-dimensional normed spaces.

Our first observation is that if $(V, \|\cdot\|)$ is a Minkowski space, it is possible to define a natural symplectic structure on the manifold of oriented lines in V, $H_1^+(V)$:

Let $\Sigma \subset V^*$ be the unit sphere for the dual norm, let $i : V \times \Sigma \to V \times V^*$ be the natural inclusion, and let $\pi_1 : V \times \Sigma \to H_1^+(V)$ be the map which sends a pair $(v, P) \in V \times \Sigma$ to the oriented line which passes through v in the direction of the Legendre transform of P. If we let ω_{can} denote the canonical symplectic form on $V \times V^*$, the symplectic form ω on $H_1^+(V)$ is defined by the equality $\pi_1^*\omega = i^*\omega_{can}$.

Theorem 4.1. *Let $(V, \|\cdot\|)$ be a Minkowski space and let $M \subset V$ be a smooth, quadratically convex hypersurface. The unit co-disc bundle for the induced Finsler metric on M and the set of all oriented lines in V which pass through the interior of M are symplectomorphic.*

In order to be able to apply lemma 1.2, we construct a natural symplectomorphism between $(H_1^+(V), \omega)$ and the cotangent of Σ provided with its canonical symplectic form ω_0.

Besides the projection $\pi_1 : V \times \Sigma \to H_1^+(V)$ defined above, let us consider the projection $\pi_2 : V \times \Sigma \to T^*\Sigma$ defined by $(v, P) \mapsto (P, v|_{[T_P\Sigma]})$. On remarking that $\pi_2^*\omega_0 = i^*\omega_{can}$, we have the following proposition:

Proposition 4.1. *If $Q_P \in T_P\Sigma$ is a covector, then $\pi_1(\pi_2^{-1}(Q_P))$ is a single point in $H_1^+(V)$ and the map $Q_P \mapsto \pi_1(\pi_2^{-1}(Q_P))$ is a symplectomorphism from $(T^*\Sigma, \omega_0)$ to $(H_1^+(V), \omega)$.*

Proof of theorem 4.1. Applying lemma 1.2 with $S := M$ and Σ, we have an anti-symplectic diffeomorphism Φ between the unit co-disc bundle of the Finsler metric on M induced from its embedding in $(V, \|\cdot\|)$ and the interior of the set

$$\{(P, v|_{[T_P\Sigma]}) : P \in \Sigma \text{ and } v \in M\} \subset T^*\Sigma.$$

Note that by composing Φ with the map that assings to each covector p_q in T^*M its opposite, $-p_q$, we obtain a symplectomorphism. The theorem now follows from the fact that the symplectomorphism between $T^*\Sigma$ and $H_1^+(V)$ takes a point $(P, v|_{[T_P\Sigma]})$ and sends it to the line passing through v in the direction of the Legendre transform of P. □

If $(V, \|\cdot\|)$ is an n-dimensional Minkowski space, we define the volume form on the manifold of oriented lines of V as the $(2n-2)$-form ω^{n-1}. Notice that this volume is invariant under translations.

Corollary 4.1 (El-Ekhtiar, [4]). . *Let $(V, \|\cdot\|)$ be a normed space of dimension n and let $M \subset V$ be a convex hypersurface. The volume of the set of lines passing through M equals the volume of M times the volume of the Euclidean unit ball of dimension $n-1$.*

Proof. If we assume that M and the unit sphere of $(V, \|\cdot\|)$ are quadratically convex, then the result follows from the previous theorem and the definition of volume in the manifold of oriented lines. An approximation argument takes care of the general case. □

5. Proof of lemmas 1.1 and 1.2

Let us start by recalling the general setup: take two compact, quadratically convex hypersurfaces, S and Σ, on a finite-dimensional vector space V and define S_Σ and Σ_S as the boundaries of the sets

$$\{(q, P_{|[T_qS]}) : q \in S, P \in \Sigma\} \subset T^*S \text{ and } \{(P, q_{|[T_P\Sigma]}) : P \in \Sigma, q \in S\} \subset T^*\Sigma.$$

Notice (see figure 1) that for every covector $(q, p) \in S_\Sigma$ there is a unique point $P \in \Sigma$ such that $P_{|[T_qS]} = p$. Indeed, the point P will be in the fold of the projection $\xi \mapsto \xi_{|[T_qS]}$ from Σ to T_q^*S. Symmetrically, for every covector $(P, Q) \in \Sigma_S$ there is a unique point $q \in S$ such that $q_{|[T_P\Sigma]} = Q$. This suggests considering the correspondence

$$\{(q, p; Q, P) \in S_\Sigma \times \Sigma_S : q_{|[T_P\Sigma]} = Q \text{ and } P_{|[T_qS]} = p\}.$$

The quadratic convexity of S and Σ implies that this correspondence is a smooth submanifold of $S_\Sigma \times \Sigma_S$ and, as such, defines a diffeomorphism $\phi : S_\Sigma \to \Sigma_S$.

Our aim will be to show that if α denotes the canonical 1-form on S_Σ and β denotes the canonical 1-form on Σ_S, then $\phi^*\beta = -\alpha + df$ for some function f on S_Σ.

Proof of lemma 1.1. In order to prove that $\phi^*\beta = -\alpha + df$, it is enough to consider the pull back of the form $\alpha+\beta$ to the correspondence and show that the resulting form is exact. Note that we can write $\alpha+\beta$ as $p \cdot dq + Q \cdot dP$ and that upon pulling back this form to the correspondence we can substitute $p \cdot dq$ by $P \cdot dq$ and $Q \cdot dP$ by $q \cdot dP$. We conclude that the pull back of $\alpha + \beta$ to the correspondence equals $P \cdot dq + q \cdot dP = d(P \cdot q)$. □

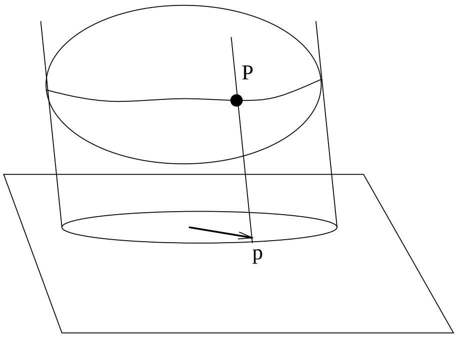

Figure 1 .

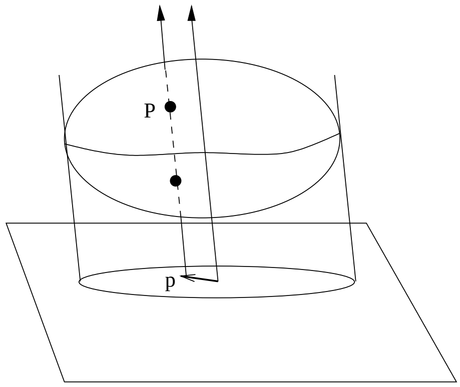

Figure 2.

We now wish to see that the map ϕ extends continuously to a homeomorphism between the sets

$$\{(q, P_{|[T_qS]}) : q \in S, P \in \Sigma\} \text{ and } \{(P, q_{|[T_P\Sigma]}) : P \in \Sigma, q \in S\}.$$

Note that if $(q,p) \in T_q^*S$ is in the domain bounded by S_Σ, then there are exactly two points P and P' on Σ such that $p = P_{|[T_qS]} = P'_{|[T_qS]}$ (see figure 2). The only difficulty in defining the extension of ϕ is in choosing one of these two points in a consistent manner. This is easier to show than to write: in figure 2 we see the hypersurface Σ and its projection onto T_q^*S. On it are marked the two preimages of the covector (q,p). The line passing through them is parallel to the kernel of the projection $\xi \mapsto \xi_{|[T_qS]}$ from V^* to T_q^*S, which is drawn in continuous trace. The kernel is oriented by the coorientation that $[T_qS]$ inherits from the coorientation of S as an embedded closed hypersurface in V. Of the two points in the preimage of (q,p) we chose the one at which the line passes from the interior to the exterior of Σ.

By analogy, for each covector (P,Q) in the domain bounded by Σ_S we know how to choose consistently one of the two points q and q' such that $Q = q_{|[T_P\Sigma]} = q'_{|[T_P\Sigma]}$. This gives us the homeomorphism we searched for. The obvious extension of the proof of lemma 1.1 shows that the the restriction of this homeomorphism to the interiors of the sets

$$\{(q, P_{|[T_qS]}) : q \in S, P \in \Sigma\} \text{ and } \{(P, q_{|[T_P\Sigma]}) : P \in \Sigma, q \in S\}.$$

is an anti-symplectic diffeomorphism.

References

[1] R. Abraham, J.E. Marsden, "Foundations of mechanics" 2nd ed, The Benjamin/Cummings Publishing Company Inc., Reading, Massachusetts, 1978.
[2] J.C. Álvarez Paiva, *The symplectic geometry of spaces of geodesics,* Ph.D. thesis, Rutgers University 1995.
[3] V.I. Arnold and A.B. Givental, *Symplectic geometry* in "Dynamical Systems IV" Encyclopaedia of Mathematical Sciences Vol. 4, V.I. Arnold and S.P. Novikov (Eds.), Springer-Verlag, Berlin Heidelberg New York, 1990.
[4] A. El-Ekhtiar, *Integral Geometry in Minkowski Space,* Ph.D. Thesis, Univ. of California, Davis (1982).
[5] R.D. Holmes and A.C. Thompson, *N-dimensional area and content in Minkowski spaces,* Pacific J. Math. **85** (1979), 77–110.
[6] J.J. Schäffer, *The self-circumference of polar convex bodies,* Arch. Math. **24** (1973), 87–90.
[7] J.J. Schäffer, "Geometry of spheres in normed spaces", Lecture Notes in Pure and Applied Mathematics. Vol. 20, Marcel Dekker, Inc. VI, New York Basel, 1976.
[8] A.C. Thompson, "Minkowski Geometry", Encyclopedia of Math. and Its Applications, Vol. 63, Cambridge Univ. Press, Cambridge, 1996.

J.C. Álvarez Paiva, Université Catholique de Louvain, Institut de Mathématique Pure et Appl., Chemin du Cyclotron 2, B–1348 Louvain–la–Neuve, Belgium.
E-mail address: `alvarez@agel.ucl.ac.be`