

## THE HILBERT METRIC AND GROMOV HYPERBOLICITY

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ABSTRACT. We give some sufficient conditions for Hilbert's metric on convex domains  $D$  to be Gromov hyperbolic. The conditions involve an intersecting chords property, which we in turn relate to the Menger curvature of triples of boundary points and, in the case the boundary is smooth, to differential geometric curvature of  $\partial D$ . In particular, the intersecting chords property and hence Gromov hyperbolicity is established for bounded, convex  $C^2$ -domains in  $\mathbb{R}^n$  with non-zero curvature.

We also give some necessary conditions for hyperbolicity: the boundary must be of class  $C^1$  and cannot contain a line segment. Furthermore we prove one statement about the asymptotic geometry of the Hilbert metric on arbitrary convex (i.e. not necessarily strictly convex) bounded domains, with an application to maps which do not increase Hilbert distances.

### INTRODUCTION

Let  $D$  be a bounded convex domain in  $\mathbb{R}^n$  and let  $h$  be the Hilbert metric, which is defined as follows. For any distinct points  $x, y \in D$ , let  $x'$  and  $y'$  be the intersections of the line through  $x$  and  $y$  with  $\partial D$  closest to  $x$  and  $y$  respectively. Then

$$h(x, y) = \log \frac{xy' \cdot x'y}{xx' \cdot yy'}$$

where  $zw$  denotes the Euclidean distance  $\|z - w\|$  between two points. For the basic properties of the distance  $h$  we refer to [Bu55] or [dlH93].

We will here give some sufficient conditions for the metric space  $(D, h)$  to be hyperbolic in the sense of Gromov. Namely, we show that a certain

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intersecting chords property implies Gromov hyperbolicity (Theorem 2.1). This intersecting chords property holds when the (Menger) curvature of any three points of the boundary of the domain is uniformly bounded from both above and below in a certain way (Corollary 1.2). Domains with  $C^2$  boundary of everywhere nonzero curvature satisfy this condition as will be proved in section 3. Beardon showed in [Be97] (see also [Be99]) that a weaker intersecting chords property holds for any bounded strictly convex domain and he used this to establish some weak hyperbolicity results for the Hilbert metric. In section 5 we prove a generalization of his results to any bounded convex domain.

It would also be interesting to understand what conditions on  $\partial D$  are necessary in order for  $(D, h)$  to be Gromov hyperbolic. For example, in section 4 we give an argument that  $\partial D$  must be of class  $C^1$  and that it cannot contain a line segment.

Some parts of the results in this paper are already known: Y. Benoist informed us that a convex domain with  $C^2$  boundary is Gromov hyperbolic if the curvature of the boundary is everywhere nonzero. Benoist has also found examples of Gromov hyperbolic Hilbert geometries whose boundaries are  $C^1$  but not  $C^2$ . In [Be99] it is mentioned that C. Bell has proved an intersecting chords theorem in an unpublished work. We have however not found the present arguments or our main results in the literature. Furthermore, we have not found the simple and attractive Proposition 1.1 and Corollary 3.5 stated or discussed anywhere, although these facts are most likely known. They should belong to ancient Greek geometry and classical differential geometry respectively.

Since the Hilbert distance can be defined in analogy with Kobayashi's pseudo-distance on complex spaces [Ko84], we would like to mention that Balogh and Bonk proved in [BB00] that the Kobayashi metric on any bounded strictly pseudoconvex domain with  $C^2$  boundary is Gromov hyperbolic. We refer to [BH99] for an exposition of this important notion of hyperbolicity.

Note that metric spaces of this type are only in exceptional cases CAT(0) (see [BH99] for the definition). Indeed, Kelly and Straus proved in [KS58] that if  $(D, h)$  is nonpositively curved in the sense of Busemann then  $D$  is an ellipsoid and hence  $(D, h)$  is the  $n$ -dimensional hyperbolic space. Compare this to the situation for Banach spaces: a Banach space is CAT(0) if and only if it is a Hilbert space. Another category of results are of the type: if  $D$  has a large (infinite, cocompact, etc.) automorphism group and smooth boundary, then it is an ellipsoid, see [SM00].

The Hilbert metric has found several applications, see [Bi57], [Li95] and [Me95] just to mention a few instances. Typically the idea is to use the contraction mapping principle to maps which do not increase Hilbert distances (e.g. affine maps).

This work was mainly done during our stay at Bielefeld University. We are grateful to this university for its hospitality and for providing such excellent working conditions. Especially we wish to thank Professor H. Abels for inviting us there.

### 1. INTERSECTING CHORDS IN CONVEX DOMAINS

From elementary school we know that if  $c_1, c_2$  are two chords in a circle and which intersect each other, then  $l_1 l'_1 = l_2 l'_2$  where  $l_1, l'_1$  and  $l_2, l'_2$  denote the respective lengths of the segments the two chords are divided into. (This follows immediately from the similarity of the associated triangles, see Fig. 1).

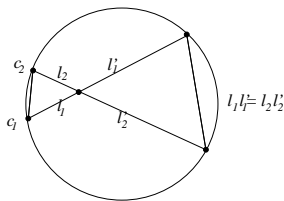


FIGURE 1  
Intersecting chords in a circle

A generalization of this fact to any bounded strictly convex domain was given by Beardon in [Be97] by an elegant argument using the Hilbert metric. He proved that if  $D$  is such a domain then for each positive  $\delta$  there is a positive number  $M = M(D, \delta)$  such that for any intersecting chords  $c_1, c_2$ , each of length at least  $\delta$ , it holds that

$$(1.1) \quad M^{-1} \leq \frac{l_1 l'_1}{l_2 l'_2} \leq M,$$

where  $l_1, l'_1$ , and  $l_2, l'_2$  denote the respective lengths of the segments the two chords are divided into.

We say that a domain satisfies the *intersecting chords property* (ICP) if (1.1) holds for *any* two intersecting chords  $c_1$  and  $c_2$ . It is easy to see that ICP may fail for a general strictly convex domain (at a curvature zero point or a "corner").

We show in this section that ICP holds for domains that satisfy a certain (non-differentiable) curvature condition. Domains with  $C^2$  boundary of nonvanishing curvature is proved to satisfy this condition in section 3.

### 1.1 INTERSECTING LINE SEGMENTS AND MENGER CURVATURE

This subsection clarifies the relation between the curvature of any triple of endpoints and the ratio as above that two intersecting line segments define.

Three distinct points  $A, B$  and  $C$  in the plane, not all on a line, lie on a unique circle. Recall that the radius of this circle is

$$(1.2) \quad R(A, B, C) = \frac{c}{2 \sin \gamma},$$

where  $c$  is the length of a side of the triangle  $ABC$  and  $\gamma$  is the opposite angle. The reciprocal of  $R$  is called the (*Menger*) *curvature* of these three points and is denoted by  $K(A, B, C)$ .

Now consider two line segments that intersect each other in one point as in Fig. 2.

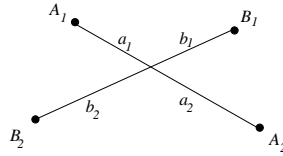


FIGURE 2  
Intersecting line segments

PROPOSITION 1.1. *In the notation above, the following equality holds:*

$$\frac{a_1 a_2}{b_1 b_2} = \frac{R(A_1, B_1, B_2) R(A_2, B_1, B_2)}{R(B_1, A_1, A_2) R(B_2, A_1, A_2)}.$$

*Proof.* Let  $\alpha_i$  be the angle between the line segments  $A_i B_j$  and  $B_1 B_2$ , and let  $\beta_i$  be the angle between  $B_i A_j$  and  $A_1 A_2$ , for  $\{i, j\} = \{1, 2\}$ . By the sine law we have

$$\begin{aligned} \frac{a_1 a_2}{b_1 b_2} &= \frac{\sin \alpha_1 \sin \alpha_2}{\sin \beta_2 \sin \beta_1} = \frac{2 \sin \alpha_1 |A_2 B_2|}{|A_1 B_1|} \frac{2 \sin \alpha_2 |A_1 B_1|}{2 \sin \beta_2 |A_2 B_2|} \frac{2 \sin \alpha_2 |A_1 B_1|}{2 \sin \beta_1} \\ &= \frac{R(A_1, B_1, B_2) R(A_2, B_1, b_2)}{R(B_1, A_1, A_2) R(B_2, A_1, A_2)}. \end{aligned}$$

□

COROLLARY 1.2. *Let  $D$  be a bounded convex domain in  $\mathbb{R}^n$ . Assume that there is a constant  $C > 0$  such that*

$$\frac{K(x, y, z)}{K(x', y', z')} \leq C$$

*for any two triples of distinct points in  $\partial D$  all lying in the same 2-dimensional plane. Then  $D$  satisfies the intersecting chords property.*

*Proof.* Any two intersecting chords define a plane and by Proposition 1.1 we have

$$\frac{a_1 a_2}{b_1 b_2} = \frac{K_{\alpha_1} K_{\alpha_2}}{K_{\beta_1} K_{\beta_2}} \leq C^2.$$

□

REMARK 1.3. In view of this subsection it is clear that ICP implies restrictions on the curvature of the boundary. We were however not able to establish the converse of Corollary 1.2.

## 1.2 CHORDS LARGER THAN $\delta$

The following proposition provides a different approach to the result in [Be97] mentioned above.

PROPOSITION 1.4. *Let  $D$  be a bounded convex domain in  $\mathbb{R}^n$ . Given  $\delta$  such that the length of any line segment contained in  $\partial D$  is bounded from above by some  $\delta' < \delta$ . Then there is a constant  $C = C(D, \delta) > 0$  such that*

$$(1.3) \quad C(D, \delta) \leq K(x, y, z) \leq \frac{1}{\delta},$$

*whenever  $x, y, z \in \partial D$  and  $xy \geq \delta$ .*

*Proof.* The angle  $\alpha(x, y, v) := \angle_y(xy, v)$  is continuous in  $x, y \in \mathbb{R}^n$  and  $v \in UT_y(\partial D)$ , the unit tangent cone at  $y$ . If  $[x, y]$  does not lie in  $\partial D$ , then  $0 < \alpha(x, y, v) < \pi$ . The set

$$S = \{(x, y, v) \in \partial D \times \partial D \times UT_y(\partial D) : xy \geq \delta\}$$

is compact. Hence there is a constant  $\alpha_0 > 0$  such that

$$(1.4) \quad \alpha_0 \leq \alpha(x, y, v) \leq \pi - \alpha_0$$

for every  $(x, y, v) \in S$ . By the definition of the tangent cone and compactness there is an  $\varepsilon > 0$  such that for any  $y, z \in \partial D$ ,  $0 < yz < \varepsilon$  there is an element  $v \in UT_y(\partial D)$  for which

$$(1.5) \quad 0 \leq \angle_y(yz, v) \leq \alpha_0/2.$$

The estimates (1.4) and (1.5) imply the existence of  $C > 0$  and the other inequality in (1.3) is trivial.  $\square$

As an immediate consequence of Propositions 1.1 and 1.4 we have :

**COROLLARY 1.5.** (Cf.[Be99]) *Let  $D$  be a bounded convex domain such that any line segment in  $\partial D$  has length less than  $\delta' < \delta$ . Then the intersecting chords property holds for any two chords each of length greater than  $\delta$ .*

## 2. HYPERBOLICITY OF HILBERT'S METRIC

Let  $(Y, d)$  be a metric space. Given two points  $z, w \in Y$ , let

$$(z|w)_y = \frac{1}{2}(d(z, y) + d(w, y) - d(z, w))$$

be their *Gromov product* relative to  $y$ . We think of  $y$  as a fixed base point. The metric space  $Y$  is *Gromov hyperbolic* (or  $\delta$ -*hyperbolic*) if there is a constant  $\delta$  such that the inequality

$$(x|z)_y \geq \min\{(x|w)_y, (w|z)_y\} - \delta$$

holds for any three points  $x, z, w$  in  $Y$ . By expanding the terms this inequality is equivalent to

$$(2.1) \quad d(x, z) + d(y, w) \leq \max\{d(x, y) + d(z, w), d(y, z) + d(x, w)\} + 2\delta.$$

**THEOREM 2.1.** *Let  $D$  be a bounded convex domain in  $\mathbb{R}^n$  satisfying the intersection chords property. Then the metric space  $(D, h)$  is Gromov hyperbolic.*

*Proof.* Suppose that the intersecting chords property holds with a constant  $M$ . Let  $y$  be a fixed reference point and consider any other three points  $x, z, w$  in  $D$ . Set  $A(u, v) = h(u, v) + h(w, y) - h(u, w) - h(v, y)$  for any two points  $u, v$ . By (2.1) we need to show that there is a constant  $\delta$  independent of  $x, z, w$  such that

$$(2.2) \quad \min\{A(x, z), A(z, x)\} \leq 2\delta.$$

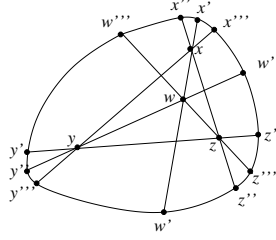


FIGURE 3

Four points

Using the definition of  $h$  and the notation in Fig. 3, we have that

$$A(x, z) = \log \left( \frac{xz'' \cdot zx''}{xx'' \cdot zz''} \frac{wy'' \cdot yw''}{ww'' \cdot yy''} \frac{xx' \cdot ww'}{xw' \cdot wx'} \frac{zz' \cdot yy'}{zy' \cdot yz'} \right) =$$

(rearranging the members of the product)

$$= \log \left( \frac{xx' \cdot xz''}{xx'' \cdot xw'} \frac{yy' \cdot yw''}{yy'' \cdot yz'} \frac{zz' \cdot zx''}{zz'' \cdot zy'} \frac{ww' \cdot wy''}{ww'' \cdot wx'} \right) \leq$$

(using  $\frac{xx'}{xx''} \leq M \frac{xz''}{xw'}$  and similar inequalities for the other fractions)

$$\leq M' + 2 \log \left( \frac{xz''}{xw'} \frac{yw''}{yz'} \frac{zx''}{zy'} \frac{wy''}{wx'} \right) \leq$$

( $y$  is fixed and  $zy', wy''$  are bounded from above and below respectively)

$$\leq M'' + 2 \log \left( \frac{xz'' \cdot zx''}{xw' \cdot wx'} \right).$$

So (2.2) is equivalent to the boundedness of

$$\min \left\{ \frac{xz'' \cdot zx''}{xw' \cdot wx'}, \frac{zx'' \cdot xz''}{zw'' \cdot wz''} \right\}.$$

from above. By symmetry we can without loss of generality assume that  $zz'' \leq xx''$ . Now we have two cases:

**Case 1:**  $xw \geq xx''$  or  $zw \geq xx''$ . If  $xw \geq zw$  (so in particular  $xw \geq xx''$ ), then

$$\begin{aligned} \frac{xz'' \cdot zx''}{xw' \cdot wx'} &\leq \frac{(xz + zz'')(zx + xx'')}{(xw)^2} \\ &\leq \frac{(xw + wz + zz'')(zw + wx + xx'')}{(xw)^2} \leq \frac{(3xw)^2}{(xw)^2} \leq 9 \end{aligned}$$

When  $zw \geq xw$ , we estimate instead the other fraction (coming from switching  $x$  and  $z$ ) in the same way.

**Case 2:**  $xw \leq xx''$  and  $zw \leq xx''$ . Considering chords at  $x$  we have

$$\frac{xz'' \cdot zx''}{xw' \cdot wx'} \leq M \frac{xx' \cdot zx''}{xx'' \cdot wx'} \leq M \frac{xx'}{wx'} \frac{(xw + wz + xx'')}{xx''} \leq 3M$$

since  $xx'' \cdot xz'' \leq M(xx' \cdot xw')$ .  $\square$

**REMARK 2.2.** Since the  $n$ -dimensional ball  $B^n$  obviously satisfies the assumption in Corollary 1.2 with  $C = 1$ , Theorem 2.1 contains the standard fact that  $(B^n, h)$ , which is Klein's model of the  $n$ -dimensional hyperbolic space, is Gromov hyperbolic.

**REMARK 2.3.** The above proof uses no compactness and therefore works in infinite dimensions provided that  $y$  lies on positive distance from the boundary. In particular, it proves that the unit ball in a Hilbert space with the Hilbert metric, which is the infinite dimensional hyperbolic space, is Gromov hyperbolic. Note however that ICP is not affinely invariant in infinite dimensions. (Kaimanovich brought this remark to our attention.)

### 3. INTERSECTING CHORDS THEOREM FOR CONVEX $C^2$ -DOMAINS

Assume that  $D$  is a bounded, convex domain in  $\mathbb{R}^n$  with  $C^2$ -smooth boundary. Let  $\rho$  be a  $C^2$ -defining function for  $D$ , that is,  $\rho$  is positive on points in  $D$ , negative outside  $\bar{D}$  and zero on  $\partial D$ . Moreover the gradient  $\nabla \rho =: \nu(x)$  is a unit vector field normal to  $\partial D$  directed inside  $D$ . The

curvature (or Weingarten) operator  $W_x: T_x\partial D \rightarrow T_x\partial D$  is by definition the directional derivative of  $\nu$  in the direction  $v$ . The second fundamental form is the bilinear form  $II_x$  on  $T_x\partial D$  given by

$$II_x(v, w) = (w, W_x(v)) = \sum_{i,j=1}^n \frac{\partial^2 \rho}{\partial x_i \partial x_j} v_i w_j.$$

The value  $II_x(u, u) =: k_x(u)$  is called the normal curvature of  $\partial D$  at  $x$  in the direction of the unit tangent vector  $u$ . We will assume that the curvature of  $\partial D$  is everywhere nonzero, meaning that  $II$  is everywhere positive definite, so there is a constant  $k_D > 0$  such that

$$(3.1) \quad k_D^{-1} \leq k_x(u) \leq k_D$$

for every  $u \in UT_x\partial D$  and  $x \in \partial D$ .

In this section we will establish:

**THEOREM 3.1.** *Let  $D$  be a bounded convex domain in  $\mathbb{R}^n$ . Suppose that the boundary  $\partial D$  is smooth of class  $C^2$  and the curvature of  $\partial D$  is everywhere nonzero. Then there is a constant  $C > 0$  such that*

$$C^{-1} \leq \frac{K(x, y, z)}{K(x', y', z')} \leq C$$

for any two triples of distinct points in  $\partial D$  all lying in the same 2-dimensional plane.

In view of Corollary 1.2 and Theorem 2.1 this implies:

**COROLLARY 3.2.** *Let  $D$  be as above. Then  $D$  has the intersecting chords property and  $(D, h)$  is Gromov hyperbolic.*

### 3.1 THE TWO DIMENSIONAL CASE

For this subsection, let  $D$  be a convex, bounded domain in  $\mathbb{R}^2$  with  $C^2$ -boundary curve  $\partial D$ . Assume in addition that the differential geometric curvature  $\kappa$  is positive (nonzero) at every point of  $\partial D$ .

**LEMMA 3.3.** *The Menger curvature  $K(x, y, z)$  of three points extends to a continuous function on  $\partial D \times \partial D \times \partial D$ . The value  $K(x, x, z)$  equals the curvature of the circle tangent to  $\partial D$  at  $x$  and passing through  $z$ , and the value  $K(x, x, x)$  equals  $\kappa(x)$ .*

*Proof.* The continuity for three distinct points is clear. When three points converge to one point on the boundary, it is a standard fact that  $K$  converges to  $\kappa$ , see [Sp78], Ch. 1 or [BG88], p. 304 or 306. When  $y_t$  converges to  $x \neq z$ , then  $d(y_t, z) \rightarrow d(x, z)$  and  $\sin \angle(y_t x, xz) \rightarrow \sin \angle(T_x \partial D, xz)$ . This proves the continuity and it is clear that the limit circle is tangent to  $\partial D$  at  $x$ .  $\square$

The idea of the proof of the following proposition was supplied to us by M. Bucher.

**PROPOSITION 3.4.** *Let  $(x, y, z)$  be a global minimum or maximum point for  $K$  on  $\partial D \times \partial D \times \partial D$ . Then  $\partial D$  contains the shortest circle arc connecting  $x, y$  and  $z$ .*

*Proof.* Recall the formula (1.2) and consider the circle in question through the three boundary points  $x, y, z$  with extremal, say maximal, radius. Denote by  $\gamma$  a shortest arc on this circle connecting these three points, and assume that  $x$  and  $z$  are the boundary points of  $\gamma$ .

In the case  $x = y = z$  there is nothing to prove. In the case the three points are all distinct, note that a point  $w$  between  $\gamma$  and  $xz$  cannot belong to  $\partial D$ , because  $R(x, w, z) > R(x, y, z)$ , (compare the angle at  $w$  with the one at  $y$ ). Moreover, note that a point  $w$  outside the circle in the half-plane defined by the line through  $x$  and  $z$  containing  $y$  cannot belong to  $\partial D$ , because either  $R(w, y, z)$  or  $R(x, y, w)$  (depending on where  $w$  lies) is greater than  $R(x, y, z)$ . Hence the arc  $\gamma$  must coincide with an arc of  $\partial D$ .

In the case  $x = y \neq z$ , no point outside the circle can lie on  $\partial D$ , again by the assumption on the maximality of the radius. On the other hand, a point  $w$  between  $\gamma$  and  $xz$  cannot belong to  $\partial D$  because  $R(x, w, z) > R(x, x, z)$ , and again we have the desired conclusion.

The case of maximal curvature can be treated analogously.  $\square$

In view of the continuity of  $\kappa$ , the following immediate consequence of Proposition 3.4 is somewhat analogous to a mean value theorem.

**COROLLARY 3.5.** *Denote by  $\kappa_{min}$  and  $\kappa_{max}$  the minimum and the maximum, respectively, of the curvature of  $\partial D$ . Then*

$$\kappa_{min} \leq K(x, y, z) \leq \kappa_{max},$$

*for any three boundary points  $x, y, z$ .*

## 3.2 THE PROOF OF THEOREM 3.1

Assume that  $D$  is as in the theorem. To save notations we will only discuss the 3-dimensional case. Each 2-dimensional plane section is Gromov hyperbolic by the above so we only need an overall bound for constants  $\delta(S)$  when  $S$  runs through all the plane sections. The intersection of  $\partial D$  with a 2-dimensional plane gives rise to a smooth planar curve  $\alpha$ , which we assume is parameterized by arclength. The constant  $\delta$  of the hyperbolicity depends on the curvature of  $\alpha$ . These curves could have an arbitrarily large curvature but we need only to bound from above (and hence from below) the ratio of the curvatures at different points of the curve. The curvature vector  $\alpha''(t)$  of  $\alpha$  at a point  $x = \alpha(t)$  lies in this plane and is orthogonal to  $\alpha'(t)$ . Thus we need to bound the ratio  $\frac{|\alpha''(t)|}{|\alpha''(s)|}$ . It is a fact (Meusnier's lemma, see [Kl78], p. 43) that

$$k_x(\alpha'(t)) = |\alpha''(t)| \cos \theta(t).$$

where  $k_x(\alpha'(t)) = II_x(\alpha'(t), \alpha'(t))$  is the normal curvature in the direction  $\alpha'(t)$  and  $\theta(t)$  is the angle between  $\alpha''(t)$  and the normal of  $\partial D$  at  $x$ . In view of the assumption 3.1 and Corollary 3.5 we therefore need to bound the ratio  $\frac{\cos \theta(s)}{\cos \theta(t)}$  independently of  $s, t$  and  $\alpha$ . Near any point  $x$  the surface  $\partial D$  is a graph of a  $C^2$  function  $z = f(x, y)$  in suitable Cartesian coordinates. Hence any small plane section  $C_\varepsilon$  is given by the equation  $f(x, y) = \varepsilon > 0$ . Expressing  $\theta$  in terms of  $f$  we arrive at the problem of bounding the ratio of the gradients  $\frac{|\nabla f(p)|}{|\nabla f(q)|}$  along the section. By rotation in the  $xy$ -plane we may assume that the  $x$ - and  $y$ -axis are along the direction of principal curvature. By developing  $f(x, y)$  into a Taylor's expansion around the origin, we obtain  $f(x, y) = \frac{1}{2}(ax^2 + by^2) + r$ , where  $r$  vanishes at  $(0, 0)$  together with all its derivatives up to second order and the principal curvatures  $a = f_{xx}(0, 0), b = f_{yy}(0, 0)$ . We conclude that  $c < \frac{|\nabla f(x, y)|}{\sqrt{x^2 + y^2}} < C$  near 0 for universal  $c, C > 0$  and thus it remains to bound the ratio  $\frac{x^2 + y^2}{x'^2 + y'^2}$  on  $C_\varepsilon$ . But this ratio is bounded in view of the estimate  $\kappa^{-1}(x^2 + y^2) < f(x, y) < \kappa(x^2 + y^2)$  for some universal  $\kappa > 0$  and that  $f(x, y) = \varepsilon$  on  $C_\varepsilon$ .  $\square$

## 4. CONSEQUENCES OF GROMOV HYPERBOLICITY FOR THE SHAPE OF THE BOUNDARY

PROPOSITION 4.1. *Let  $D$  be a bounded convex domain in  $\mathbb{R}^n$  and let  $h$  be a Hilbert metric on  $D$ . If  $h$  is Gromov hyperbolic then the boundary  $\partial D$  is strictly convex, that is, it does not contain a (nondegenerate) line segment.*

This can be proven following the proof of N. Ivanov [Iv97] of Masur-Wolf's theorem [MW95] that the Teichmüller spaces (genus  $\geq 2$ ) are not Gromov hyperbolic. The proof makes use of Gromov's exponential divergence criterion, see [BH99] p. 412.

THEOREM 4.2. *Let  $D$  be a bounded convex domain in  $\mathbb{R}^n$  and let  $h$  be the Hilbert metric on  $D$ . If  $h$  is Gromov hyperbolic then the boundary  $\partial D$  is smooth of class  $C^1$ .*

*Proof.* *2-dimensional case.* First, by the previous result,  $D$  is strictly convex. Let  $y = f(x)$ ,  $x \in (-a, a)$  be an equation of  $\partial D$  near some point. Then  $f$  is strictly convex and hence one-sided derivatives  $f'_-(x)$ ,  $f'_+(x)$  exist and are strictly increasing on  $(\varepsilon, \varepsilon)$ , [RV73], § 11.

We prove that  $f'_-(0) = f'_+(0)$ . Suppose not, then by choosing appropriate Cartesian coordinates we may assume that  $f'_-(0) < 0$  and  $f'_+(0) > 0$ . For each sufficiently small  $\varepsilon$  build an ideal triangle  $\Delta = \Delta(\varepsilon)$  in  $D$  with one vertex 0 and two other vertices corresponding to the intersection of the line  $y = \varepsilon$  with  $\partial D$ . We assert that the slimness of  $\Delta(\varepsilon)$  tends to  $\infty$  when  $\varepsilon$  tends to zero. Namely we show that the Hilbert distance between the point  $P = (0, \varepsilon)$  and any point  $Q$  of the side  $[0, B]$  tends to  $\infty$ . Let  $f'_+(0) = \tan \alpha$ ,  $0 < \alpha < \pi/2$ . Let  $x_1 < x_2$  be the points such that  $f(x_1) = \varepsilon$  and  $f'_+(0)x_2 = \varepsilon$ . Then

$$PQ \geq \varepsilon \cos \alpha = f(x_1) \cos \alpha.$$

Let  $O, R$  be the intersection points of the line  $PQ$  with  $\partial D$ . We have therefore

$$QR \leq x_2 - x_1 = \frac{f(x_1)}{f'_+(0)} - x_1 = \frac{f(x_1) - f'_+(0)x_1}{f'_+(0)}$$

and hence combining the last two inequalities

$$\begin{aligned} \frac{PQ}{QR} &\geq \frac{f'_+(0)f(x_1) \cos \alpha}{f(x_1) - f'_+(0)x_1} \\ &= \frac{f'_+(0) \cos \alpha}{1 - f'_+(0)\frac{x_1}{f(x_1)}} \rightarrow \infty \text{ when } x_1 \rightarrow 0. \end{aligned}$$

It follows that

$$h(P, Q) = \ln \left( 1 + \frac{PQ}{OP} \right) \left( 1 + \frac{PQ}{QR} \right) \rightarrow \infty \text{ when } x_1 \rightarrow 0$$

and hence the slimness of  $\Delta(\varepsilon)$  tends to  $\infty$  when  $\varepsilon$  tends to zero.

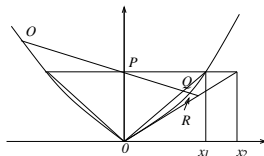


FIGURE 4  
Hyperbolicity implies  $C^1$

It remains to show that  $f'$  is continuous. By [RV73], §14 we have

$$\lim_{x \rightarrow x_0+} f'_+(x) = f'_+(x_0),$$

$$\lim_{x \rightarrow x_0-} f'_-(x) = f'_-(x_0).$$

From this we conclude that  $f'_+$  is continuous in  $x_0$  since  $f'_+(x_0) = f'_-(x_0)$ . But  $f'(x_0) = f'_+(x_0)$  hence  $f'$  is also continuous at  $x_0$ .

*n-dimensional case.* By the known result that if  $f$  is a differentiable convex function defined on an open convex set  $S$  in  $\mathbb{R}^{n+1}$ , then it is  $C^1$  on  $S$ , see for example [RV73]. Let  $D$  be a bounded convex domain in  $\mathbb{R}^{n+1}$ ,  $n \geq 2$ . It is enough to prove that  $\partial D$  is differentiable at any point. Given a point  $p \in \partial D$ , we can choose the coordinate axis of  $\mathbb{R}^{n+1}$  so that the origin  $O$  of the coordinates is at  $p$ , all of  $D$  lies in the half-space  $x_0 \geq 0$  and in a neighbourhood of  $p$  the surface  $\partial D$  can be represented as a graph of a nonpositive convex function  $x_0 = f(x_1, x_2, \dots, x_n)$ ,  $x = (x_1, x_2, \dots, x_n)$ ,  $f(0) = 0$ . Considering the 2-dimensional sections in the planes  $x_0, x_i, i = 1, \dots, n$  we obtain that the partial derivatives of  $f$  at 0 exist and  $f_{x_i}(0) = 0$ ,  $i = 1, \dots, n$ . We have to prove that for each  $\varepsilon > 0$  there is a neighbourhood  $U_\varepsilon$  of 0 such that  $f(x) < \varepsilon|x|$  in this neighbourhood. But in view of  $f_{x_i}(0) = 0$ ,  $i = 1, \dots, n$ , we have  $f(0, \dots, 0, x_i, 0, \dots, 0) < \varepsilon|x_i|$  for sufficiently small  $x_i$  and hence by convexity  $f(x) < \varepsilon|x|$  for sufficiently small  $|x|$ .  $\square$

REMARK 4.3. The following is announced in [B00]: If a strictly convex domain  $D$  is divisible, that is, it admits a proper cocompact group of isometries  $\Gamma$ , then  $D$  is Gromov hyperbolic if and only if  $\partial D$  is  $C^1$ . Our Theorem 4.2 shows that that in the implication (Gromov hyperbolicity + divisibility  $\Rightarrow C^1$ ) the condition of divisibility is superfluous.

## 5. NON-STRICTLY CONVEX DOMAINS

This section owes much of its existence to [Be97] and [Be99]. We prove using a different argument certain extensions to arbitrary convex bounded domains of some of the results obtained in those papers.

LEMMA 5.1. *Let  $D$  be a bounded convex domain in  $\mathbb{R}^n$ . Let  $\{x_n\}, \{y_n\}$  be two sequences of points in  $D$ . Assume that  $x_n \rightarrow \bar{x} \in \partial D$ ,  $y_n \rightarrow \bar{y} \in \bar{D}$  and  $[\bar{x}, \bar{y}] \not\subseteq \partial D$ . Let  $x'_n$  and  $y'_n$  denote the endpoints of the chord through  $x_n$  and  $y_n$  as usual. Then  $x'_n$  converges to  $\bar{x}$  and  $y'_n$  converges to the endpoint  $\bar{y}'$  of the chord defined by  $\bar{x}$  and  $\bar{y}$  different from  $\bar{x}$ .*

*Proof.* Compare with Lemma 5.3. in [Be97]. Every limit point of chord endpoints must belong to the line through  $\bar{x}$  and  $\bar{y}$ . In addition, in the case of  $x'_n$  for example, any limit point must lie in the halfline from  $\bar{x}$  not containing  $\bar{y}$ . At the same time every limit point must belong to the boundary of  $D$ , and the statement follows since the line through  $\bar{x}$  and  $\bar{y}$  intersects  $\partial D$  only in  $\bar{x}$  and  $\bar{y}'$ .  $\square$

THEOREM 5.2. *Let  $D$  be a bounded convex domain. Let  $\{x_n\}$  and  $\{z_n\}$  be two sequences of points in  $D$ . Assume that  $x_n \rightarrow \bar{x} \in \partial D$ ,  $z_n \rightarrow \bar{z} \in \partial D$  and  $[\bar{x}, \bar{z}] \not\subseteq \partial D$ . Then there is a constant  $K = K(\bar{x}, \bar{z})$  such that for the Gromov product  $(x_n | z_n)_y$  in Hilbert distances relative to some fixed point  $y$  in  $D$  we have*

$$\limsup_{n \rightarrow \infty} (x_n | z_n)_y \leq K.$$

*Proof.* By Lemma 5.1, the endpoints of the chords through  $x_n$  and  $z_n$  converge to  $\bar{x}$  and  $\bar{z}$ .

Since  $[\bar{x}, \bar{z}]$  is not contained in the boundary of  $D$ , there are small compact neighbourhoods  $\bar{U}_x$  and  $\bar{U}_z$  of  $\bar{x}$  and  $\bar{z}$  respectively, in  $\partial D$ , such that every chord with endpoints in  $\bar{U}_x$  and  $\bar{U}_z$  is contained in  $D$ . In

particular the Euclidean midpoint of every such chord lies inside  $D$  and by compactness there is an upper bound  $K$  on  $h(y, w)$ , where  $w$  is the midpoint of such a chord.

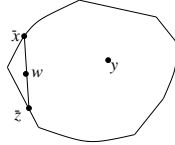


FIGURE 5  
Partial hyperbolicity

Consider three points  $x, y, z$  and a point  $w$  on a (minimizing) geodesic segment  $[x, z]$  in a (geodesic) metric space  $(Y, d)$ . Then

$$\begin{aligned} (x|z)_y &= \frac{1}{2}(d(x, y) + d(z, y) - d(x, z)) \\ &= \frac{1}{2}(d(x, y) + d(z, y) - d(x, w) - d(w, z)) \\ &\leq \frac{1}{2}(d(y, w) + d(y, w)) = d(y, w) \end{aligned}$$

by the triangle inequality. It follows from this estimate and the above considerations that eventually

$$(x_n|z_n)_y \leq K.$$

□

REMARK 5.3. The content of Theorem 5.2 is that  $(D, h)$  satisfies a weak notion of hyperbolicity. This property should be compared with Gromov hyperbolicity, especially with the fact that for Gromov hyperbolic spaces, two sequences converge to the same point of the boundary if and only if their Gromov product tends to infinity. Theorem 5.2 can be used as in [Ka99], Theorem 8, to the study of random walks on the automorphism group of  $D$ , and it is also likely to be useful for analyzing commuting nonexpanding maps or isometries of  $(D, h)$ .

REMARK 5.4. We suggest that a similar statement might hold for the classical Teichmüller spaces and perhaps also for more general Kobayashi hyperbolic complex spaces. Hilbert geodesic rays from a point  $y$  that

terminate on a line segment contained in the boundary may correspond to the Teichmüller geodesic rays defined by Jenkins-Strebel differentials that H. Masur considered when demonstrating the failure of CAT(0) for the Teichmüller space of Riemann surfaces of genus  $g \geq 2$ . The complement of the union of all line segments in the boundary  $\partial D$  may correspond to the uniquely ergodic foliation points on the Thurston boundary of Teichmüller space.

Using the arguments in [Ka99], see Proposition 5.1 of that paper, we obtain the following result as an application of Theorem 5.2:

**THEOREM 5.5.** *Let  $D$  be a bounded convex domain and  $\varphi : D \rightarrow D$  be a map which does not increase Hilbert distances. Then either the orbit  $\{\varphi^n(y)\}_{n=1}^{\infty}$  is bounded or there is a limit point  $\bar{y}$  of the orbit such that for any other limit point  $\bar{x}$  of the orbit it holds that  $[\bar{x}, \bar{y}] \subset \partial D$ .*

#### REFERENCES

- [BB00] BALOGH, Z. M. and BONK, M. Gromov hyperbolicity and the Kobayashi metric on strictly pseudoconvex domains. *Comment. Math. Helv.* 75 (2000), 504–533.
- [Be97] BEARDON, A. F. The dynamics of contractions. *Erg. Th. & Dyn. Sys.* 17(6) (1997), 1257–1266
- [Be99] BEARDON, A. F. The Klein, Hilbert and Poincaré metrics of a domain. *J. Comput. Appl. Math.*, 105(1-2) (1999), 155–162. Continued fractions and geometric function theory (CONFUN) (Trondheim, 1997).
- [B00] BENOIST, Y. Convexes divisibles, <http://www.dma.ens.fr/~benoist/prepublication.html>
- [BG88] BERGER M. and GOSTIAUX, B. *Differential geometry: manifolds, curves, and surfaces*. Translated from the French by Silvio Levy Springer-Verlag (New York), 1988.
- [Bi57] BIRKHOFF, G. Extensions of Jentzsch's theorem. *Trans. Amer. Math. Soc.* 85 (1995), 219–227
- [BH99] BRIDSON M. and HAFLIGER, A. *Metric spaces of non-positive curvature*. Grundlehren der Math. Wissenschaften, 319 Springer-Verlag (Berlin), 1999.
- [Bu55] BUSEMANN, H. *The Geometry of Geodesics*. Academic Press Inc. (New York), 1955.
- [dlH93] DE LA HARPE, P. On Hilbert's metric for simplices. In: *Geometric Group Theory, Vol. 1 (Sussex, 1991)* pp. 97–119. Cambridge Univ. Press (Cambridge), 1993.

- [Iv97] IVANOV, V. I. A short proof of Gromov non-hyperbolicity of Teichmüller spaces. Preprint, Michigan State University, 1997.
- [Ka99] KARLSSON, A. Nonexpanding maps and Busemann functions. To appear in *Ergod. Th. & Dyn. Sys.*, 1999.
- [Kl78] KLINGENBERG, W. *A course in differential geometry*. Translated from the German by David Hoffman. Graduate Texts in Mathematics, Vol. 51. Springer-Verlag (New York), 1978.
- [Ko84] KOBAYASHI, S. Projectively invariant distances for affine and projective structures. In: *Differential geometry (Warsaw, 1979)*, pp. 127–152. PWN, Warsaw, 1984.
- [KS58] KELLY, P. and STRAUS, E. Curvature in Hilbert geometries. *Pacific J. Math.* 8 (1958), 119–125.
- [Li95] LIVERANI, C. Decay of correlations *Ann. of Math. (2)* 142 (1995), 239–301.
- [MW95] MASUR, H. and WOLF, M. Teichmüller space is not Gromov hyperbolic. *Ann. Acad. Sci. Fenn. Ser. A I Math.* 20(2) (1995), 259–267.
- [Me95] METZ, V. Hilbert's projective metric on cones of Dirichlet forms. *J. Funct. Anal.* 127 (1995), 438–455.
- [RV73] ROBERTS, A. W. and VARBERG, D. E. *Convex functions*. Pure and Applied Mathematics, Vol. 57. Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers] (New York-London), 1973.
- [SM00] SOCIÉ-MÉTHOU, E. *Comportements asymptotiques et rigidités en géométries de Hilbert*. Thesis, IRMA-Strasbourg, 2000.
- [Sp79] SPIVAK, M. *A Comprehensive Introduction to Differential Geometry*. Vol. 2., Publish or Perish, Inc., 1979.

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