ON THE PERIMETER AND AREA OF THE UNIT DISC

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1. Introduction

In the preamble to his fourth problem (presented at the International Mathematical Congress in Paris in 1900) Hilbert suggested a thorough examination of geometries that "stand next to Euclidean geometry" in the sense that they satisfy all the axioms of Euclidean geometry except one. In non-euclidean geometries the axiom that is usually taken to fail is the famous parallel postulate. This leads to the relatively well-known hyperbolic and elliptic geometries. The significance of these is that, like Euclidean geometry, they are homogeneous (all points have the same status) and isotropic (all directions have the same status).

Another type of geometry that "stands next to Euclidean geometry" is the geometry of normed spaces. Here translating a line segment does not change its length, but the axiom that states that two triangles with equal corresponding sides are congruent no longer holds.

In this article we survey some of the most basic results on the geometry of unit discs in two-dimensional normed spaces, while adding a few results and some new proofs of our own. These results answer simple questions such as: what is the perimeter of the unit disc, what is its area, and what is the relations between these two quantities?

2. Minkowski planes and duality

A *Minkowski plane* is a two-dimensional affine space provided with a metric that is invariant under translations. Neglecting the difference between an affine space and a vector space (the choice of a zero vector), we may consider a Minkowski plane as a pair $(\mathbb{R}^2, d)$, where d is a distance function satisfying $d(x+v, y+v) = d(x,y)$ for all triples of vectors x, y, and v.

If $(\mathbb{R}^2, d)$ is a Minkowski plane, it is easy to see that the function $\|x\| := d(x, 0)$ is a *norm:*

(1) $\|x\| \geq 0$ with equality if and only if x is the zero vector.
(2) $\|tx\| = |t|\|x\|$;

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(3) $\|x+y\| \leq \|x\| + \|y\|$.

Conversely, if $\|\cdot\|$ is a norm on $\mathbb{R}^2$, then the function $d(x,y) := \|x-y\|$ defines a translation-invariant metric on the plane.

As a result of this remark, any norm in the plane gives rise to a Minkowski plane and vice-versa. For example, the classical p-norms,

$$\|x\|_p := \left(|x_1|^p + |x_2|^p\right)^{1/p}, \quad p \geq 1,$$

and the *max* or *sup* norm,

$$\|x\|_\infty := \max\{|x_1|, |x_2|\},$$

provide us with some ready examples of Minkowski planes.

Exercise 2.1. Consider the 1-norm, or *taxicab norm,* on the plane, $\|x\| := |x_1| + |x_2|$, and its corresponding distance function $d(x,y) = \|x-y\|$. Construct two non-congruent triangles with corresponding sides of length 1, 1, and 2.

In his pioneering work on the geometry of numbers, Minkowski realized that it is often better to consider the unit disc

$$D := \{x : \|x\| \leq 1\}$$

as the starting point for the investigation of normed planes and Minkowski geometries.

Exercise 2.2. Show that the definition of a norm implies that the unit disc

(1) is a closed, bounded set with 0 as an interior point;
(2) is symmetric with respect to 0 (*i.e.,* if $x \in D$ then $-x \in D$);
(3) is convex.

Conversely, show that if K is a set satisfying the properties above, then the function

$$\|x\|_K := \min\{\lambda : x \in \lambda K\}$$

is a norm for which K is the unit disc.

The last condition is crucial for the triangle inequality and this is why we cannot allow $p < 1$ in the definition of $\|x\|_p$.

Exercise 2.3. Piet Hein's *supercircle* is the unit circle for the p-norm with $p = 5/2$. According to Martin Gardner (see [9]) this is a 'squared circle' in the sense that it is artistically midway between a circle and a square. Graph the supercircle and see if you agree with Gardner's description.

Now we have a great variety of norms on the plane. We only need to consider a symmetric convex body K and obtain the norm it generates. For example we may take K to be a regular $2n$-gon. Because of the invariance under translations the associated Minkowski geometries are *homogeneous* —

every point behaves like every other point. However, these geometries are *not* isotropic except in the case where the norm comes from an inner product (*i.e.,* the unit disc is an ellipse).

Exercise 2.4. Translations, the identity map, I, and symmetry about the origin, $-I$, are isometries for any Minkowski plane. Show that in general these are the only ones by constructing a convex set K such that the normed space $(\mathbb{R}^2, \|x\|_K)$ has this minimal set of isometries.

A unique feature of Minkowski geometry is the notion of duality. The *dual space* of $\mathbb{R}^2$, denoted by $(\mathbb{R}^2)^*$, is the space of all linear functions from $\mathbb{R}^2$ to $\mathbb{R}$. If $p \in \mathbb{R}^{2*}$ and $q \in \mathbb{R}^2$, we denote the pairing of p and q by $p \cdot q$. A norm $\| \cdot \|$ on $\mathbb{R}^2$ induces a *dual norm* $\| \cdot \|^*$ on $\mathbb{R}^{2*}$ by the formula

$$\|p\|^* := \sup\{|p \cdot q| : \|q\| \leq 1\}$$

If K is the unit disc in $(\mathbb{R}^2, \| \cdot \|)$, we denote the unit disc of $(\mathbb{R}^{2*}, \| \cdot \|^*)$ by K°. It is usual to call K° the *polar* of K.

Exercise 2.5. Show that if K is the unit disc of a Minkowski plane, then $(K^\circ)^\circ = K$.

In practice, to draw the polar of a unit disc K we identify $\mathbb{R}^2$ and $\mathbb{R}^{2*}$ by using the standard basis and plot all the points p for which the line $p \cdot x = 1$ is tangent to the boundary of K. We precise that by *tangent* we mean that the line intersects the boundary of K, but not its interior. Notice that it is not necessary for the boundary of K to be differentiable in order to talk about tangent lines in this sense.

Exercise 2.6. Show that the taxi cab and the max norm are dual to each other. In general, a celebrated theorem of Minkowski states that the p-norm and the q-norm are dual to each other if and only if $p^{-1} + q^{-1} = 1$.

A more complete account of Minkowski planes can be found in Thompson's book [22] and in the survey [15] by Martini, Swanepoel, and Weiss. Section 7.4 of this last paper served as a motivation for the present, more leisurely, account of the geometry of unit discs in Minkowski planes.

3. Basic geometry of Minkowski planes

Having dealt with the fundamental ideas, in this section we look at three topics from Minkowski geometry. These are (i) the perimeter of the unit disc; (ii) the notion of normality (perpendicularity); and (iii) the area of the unit disc.

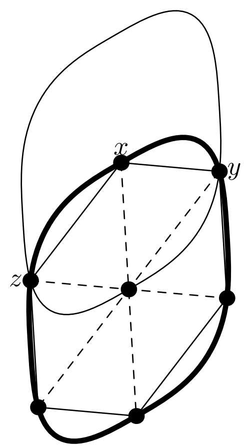

Fig. 1. Inscribing an affine regular hexagon.

3.1. **Perimeter of the disc.** On any metric space (X, d) it is possible to define the length of a curve as the supremum over all partitions $\{x_0, x_1, \ldots, x_n\}$ of the quantity $\sum_{i=1}^{n} d(x_i, x_{i-1})$. On a Minkowski plane, we may also compute the length of a smooth parameterized curve, $\gamma(t)$, $a \leq t \leq b$, as the integral

$$\int_b^a \|\dot{\gamma}(t)\| dt.$$

Given a regular curve, we may always parameterize it by Minkowski arc-length s. In this case, $\dot{\gamma}(s)$ is a (Minkowski) unit vector. If K is a convex set in a Minkowski plane, we let ∂K denote its boundary and $\ell(\partial K)$ its Minkowski length.

The most obvious curve to consider is the boundary of the unit disc. We will call this the *unit circle.* In the Euclidean case, the length of this curve is the most precisely calculated of all transcendental numbers. In what follows we shall *only* consider the length of the unit circle relative to its *own* norm.

Exercise 3.1. Show that if the unit disc is a parallelogram, its perimeter is eight. If the unit disc is a regular hexagon, its perimeter is six. Construct a family of hexagons H_t, $(0 \leq t \leq 1)$, such that when considered as unit discs we have $\ell(\partial H_t) = (6 + 2t)$. Note that when $t = 1$ the hexagon becomes a parallelogram.

There are two basic theorems about the perimeter of the unit disc in a Minkowski plane. The first is relatively well known, the second much less so.

Theorem 3.1 (Gołąb, [10])**.** *If D is the unit disc of a Minkowski plane then*

$$6 \leq \ell(\partial D) \leq 8$$

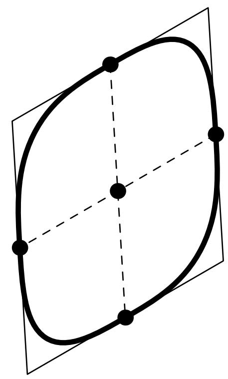

Fig. 2. A minimal circumscribing parallelogram.

with equality on the left if and only if D is an affine regular hexagon and equality on the right if and only if D is a parallelogram.

We may say then that in Minkowski geometry the value of "π" (the ratio of the circumference to the diameter of a disc) can take any value between 3 and 4.

Sketch of the proof. To show the lower bound one inscribes in D an affine regular hexagon by the construction illustrated in figure 1: translate a copy of the unit circle by a (generic) unit vector x and mark the two intersection points, y and z. The hexagon is given by the convex hull of x, y, z, and their opposites. Each side of this hexagon is a translate of a unit vector (illustrated by the dashed lines) and, therefore, the length of each side is equal to one.

To show the upper bound one circumscribes to D a parallelogram P in such a way that the points of tangency bisect its sides. One way of doing this is to take P to be a minimal circumscribing parallelogram: the area of P is less than or equal to the area of any other parallelogram containing D. In this case the sides of the parallelogram are easily seen to have length two (Fig. 2).

The hard part of the proof is to show that the bounds are attained *only* for the affine regular hexagon and the parallelogram. References for this part are Schäffer [20], Petty [17], and Thompson [22]. □

Theorem 3.2 (Schäffer, [21]). *If D is the unit disc for a Minkowski plane and if D° is the unit disc in the dual plane, then*

$$\ell(\partial D) = \ell(\partial D^\circ).$$

A new proof of this result will be given in the last section.

3.2. **Normality in Minkowski planes.** One of the ideas that plays a fundamental role in Euclidean geometry is that of orthogonality. Not only does this concept occur in Euclid's axioms themselves, but also in many of the basic theorems (for example, in that staple of high school geometry, Pythagoras' theorem). One of the underlying themes in Minkowski geometry is to look for suitable substitutes for this notion. In what follows we shall be concerned with one of these in particular.

In Euclidean geometry a tangent to a circle is perpendicular to the radius that joins the center to the point of tangency. We make this into a definition. The idea goes back at least to Carathèodory but it is hard to give a precise reference. The most frequently cited reference for the definition is Birkhoff [5], but see also James [13].

Definition 3.1. If $(\mathbb{R}^2, \|.\|)$ is a Minkowski plane, we say that q is *normal* to y and write $q \dashv y$ if $\|q + \alpha y\| \geq \|q\|$ for all $\alpha \in \mathbb{R}$.

In thinking about this definition one may suppose that $\|q\| = 1$ so that q lies on ∂D. The condition then says that the line $q + \alpha y$ is tangent to D at q (Fig. 3). In particular, if ∂D admits a differentiable parameterization $\gamma(s)$, then $\gamma(s) \dashv \dot{\gamma}(s)$.

An important remark is that normality is symmetric *only* for a special class of normed planes:

Definition 3.2. If $(\mathbb{R}^2, \|\cdot\|)$ is a normed plane for which normality is symmetric, then the boundary of its unit disc is said to be a *Radon curve.*

Exercise 3.2. This exercise presents everything the reader should know about Radon curves to understand the theorems and proofs in the rest of the paper.

(1) Show that the regular hexagon is a Radon curve, but that the regular octagon is not.
(2) Show that the image of a Radon curve by an invertible linear transformation is also a Radon curve.
(3) Show that the polar of a Radon curve is also a Radon curve.
(4) Let D be the unit disc of a normed plane such that ∂D admits a differentiable parameterization $\gamma(s)$ by Minkowski arc-length. Show that ∂D is a Radon curve if and only if the function $s \mapsto \det(\gamma(s), \dot{\gamma}(s))$ is constant.
(5) According to Martini, Swanepoel, and Weiss (see [15]), a centrally-symmetric, convex curve is *equiframed* if every one of its points is a point of tangency for some minimal circumscribing parallelogram. Show that every Radon curve is equiframed, and that every regular equiframed curve is a Radon curve. However, show that the regular octagon is equiframed.

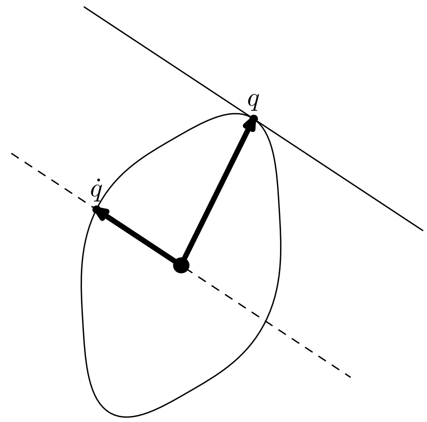

Fig. 3. The vector q is normal to $\dot{q}$.

Although normality is not generally symmetric on any Minkowski plane, there is always at least one pair of unit vectors that are mutually normal. The proof of this is quite simple in the case where the unit circle admits a differentiable parameterization by Minkowski arc-length $\gamma(s)$: let u be a value of the parameter s for which the function $\Delta(s) := \det(\gamma(s), \dot{\gamma}(s))$ reaches its maximum. Since $0 = \Delta'(u) = \det(\gamma(u), \ddot{\gamma}(u))$, we have that $\dot{\gamma}(u) \dashv \gamma(u)$ as well as $\gamma(u) \dashv \dot{\gamma}(u)$.

When ∂D is not a regular curve, we may either follow Day [7] and consider parallelograms of minimal area circumscribing D or follow Taylor [24] and consider inscribed parallelograms of maximal area. In the first case the sides of the parallelogram are mutually normal and in the second two adjacent vertices are mutually normal. Note that the polar of a minimal parallelogram circumscribing D is a maximal parallelogram inscribed in D° and vice-versa.

For the final part of this discussion of normality we will assume that ∂D and ∂D° are smooth.

Definition 3.3. If $q \in \partial D$ then the *Legendre transform* of q is that unique point $\mathcal{L}(q) \in \partial D^\circ$ such that the line $\mathcal{L}(q) \cdot x = 1$ is tangent to D at q.

The null space of $\mathcal{L}(q)$ consists of those vectors y such that $q \dashv y$. In particular, if $\gamma(s)$ is a parameterization of the unit circle, then $\mathcal{L}(\gamma(s)) \cdot \dot{\gamma}(s) = 0$.

Exercise 3.3. This exercise presents two important properties of the Legendre transform that will be used in the proof of proposition 3.1.

(1) Show that the Legendre transform is an involution: $\mathcal{L}(\mathcal{L}(q)) = q$ for any $q \in \partial D$.
(2) If v, w are unit vectors that form a positively oriented basis of $\mathbb{R}^2$, then $\mathcal{L}(v)$, $\mathcal{L}(w)$ form a positively oriented basis of $\mathbb{R}^{2*}$.

To end this section we uncover a simple relationship between normality and the Legendre transform which will play a crucial in our proof of Schäffer's theorem.

Proposition 3.1. *Let q be a unit vector and let $\dot{q}$ denote the unique unit vector such that $q \dashv \dot{q}$ and $(q, \dot{q})$ is a positively oriented basis for $\mathbb{R}^2$. If $q \in \partial D$ and if $p := \mathcal{L}(\dot{q})$, then $\mathcal{L}(\dot{p}) = -q$.*

Proof. By the definition of the Legendre transform, we have that

$$\mathcal{L}(p) \cdot \dot{p} = 0 \quad \text{and} \quad \mathcal{L}(q) \cdot \dot{q} = 0.$$

Applying $\mathcal{L}$ to both sides of the equality $p := \mathcal{L}(\dot{q})$, we obtain that $\mathcal{L}(p) = \dot{q}$. So the equation on the left above is $\dot{p} \cdot \dot{q} = 0$. Comparing this with the equation on the right above we see that $\mathcal{L}(q) = \pm\dot{p}$ and hence $q = \pm\mathcal{L}(\dot{p})$. Finally, since both $(q, \dot{q})$ and $(\mathcal{L}(p), \mathcal{L}(\dot{p})) = (\dot{q}, \mathcal{L}(\dot{p}))$ are positively oriented bases we must take the negative sign. □

3.3. **Areas in Minkowski planes.** In contrast with the measurement of lengths, there is nothing in the definition of a Minkowski plane that will fix a canonical and uncontestable way of measuring areas. Nevertheless, a growing body of work in convex, integral, and Finsler geometry shows that not only a theory of areas and volumes in finite-dimensional normed spaces is possible, but that it provides a common framework for some of the deepest theorems and problems in those fields (see the book [22], the recent survey [2], and the references therein).

The first significant remark in approaching the problem of measuring areas in a Minkowski plane is that since translations are isometries, the area of a region should be invariant under translations. If we also require — as it is natural to do — that the area of a compact set be finite and that the area of an open set be positive, then a deep theorem of Haar (see [22], pp. 37) implies that the area is a positive multiple of the Lebesgue measure. Thus, all that is required is a suitable normalization of the Lebesgue measure.

Since we would like an isometry between two Minkowski planes to be area preserving, this normalization cannot be arbitrary. For example, if our unit disc D is an ellipse, then our space is isometric to the Euclidean plane, and we must assign to the ellipse the area π. Note that it doesn't matter how large or small we draw D on the page, its area is π. Thus, a "suitable normalization" means that we assign to the unit disc D in our Minkowski plane some number that is an affine invariant of D that takes the value π on ellipses. While there are infinitely many ways of doing this, we shall concentrate on four normalizations that have appeared, sometimes implicitly, in different contexts in convex and differential geometry.

The first normalization, proposed by Busemann, is to assign the area π to the unit disc D of any Minkowski plane. The advantage is simplicity, the disadvantage is that it does not reflect the differences in shape between the unit discs of different Minkowski planes. We will call this the *Busemann definition of area* and denote it by μ_b.

A second normalization consists in prescribing the area of the unit disc to be $1/\pi$ times the volume of $D \times D^\circ$ in the (symplectic) space $\mathbb{R}^2 \times \mathbb{R}^{2*}$. We recall that the canonical volume form on $\mathbb{R}^2 \times \mathbb{R}^{2*}$ is defined by taking a basis e_1, e_2 in $\mathbb{R}^2$, its dual basis e_1^*, e_2^* in $\mathbb{R}^{2*}$, and setting the volume of the parallelotope $e_1 \wedge e_2 \wedge e_1^* \wedge e_2^*$ to be one. It is easy to check that the choice of basis in $\mathbb{R}^2$ is of no importance. This is the *Holmes-Thompson definition of area* introduced in [12], and which we will denote by μ_{ht}. In what follows, it will be convenient to express the Lebesgue measure on $\mathbb{R}^2$ by λ and its dual measure on $\mathbb{R}^{2*}$ by λ^*. Thus, the symplectic volume of $D \times D^\circ$ can be written as $\lambda(D)\lambda^*(D^\circ)$.

The third normalization we shall consider is obtained by assigning area 2 to any maximal inscribed parallelogram in D. If we assign area 4 to any minimal circumscribing parallelogram of D, then we obtain a fourth possible normalization. These two notions of area were introduced by Gromov as mass and mass*, respectively, in his landmark paper [11]. Convex geometers will recognize mass* as the *Benson definition of area* (see [3, 4], and [23]). In what follows we will denote mass by μ_m, and mass* by μ_{m*}.

Exercise 3.4. Show that if $(\mathbb{R}^2, \|.\|)$ is a normed plane with unit disc D then

$$\mu_{ht}(D)\mu_b(D) = \lambda(D)\lambda(D^\circ) = \mu_m(D)\mu_{m*}(D^\circ).$$

An interesting feature of definitions of areas on Minkowski planes (or spaces) is that they come in dual pairs: If μ is a definition of area and $\mu(v_1 \wedge v_2; \|\cdot\|)$ denotes the μ-area of the parallelogram $v_1 \wedge v_2$ in the normed plane $(R^2, \|\cdot\|)$, we define the *dual definition of area* by the equality

$$\mu^*(v_1 \wedge v_2; \|\cdot\|) = \mu(\xi_1 \wedge \xi_2; \|\cdot\|^*)^{-1},$$

where $\xi_1 \wedge \xi_2 \cdot v_1 \wedge v_2 = 1$.

Exercise 3.5. Show that the area definitions of Busemann and Holmes-Thompson, as well as mass and mass*, are dual to each other.

It is not clear what role duality plays in the more detailed study of the different definitions of volume and area on normed spaces, but the following easy remark will be used later on in the paper:

Exercise 3.6. Show that if μ, μ^*, ν, and ν^* are two dual pairs of area definitions for Minkowski planes, then $\mu \geq \nu$ if and only if $\mu^* \leq \nu^*$.

4. Bounding the area of the unit disc

Gołąb's theorem gives precise bounds for the perimeter of the unit disc. In this section we will begin the investigation of bounds for its area using the Holmes-Thompson, mass, and mass* definitions. Note that since μ_b is constant for all unit discs the question of bounds for the Busemann definition is trivial.

For the Holmes-Thompson definition the question of bounds is equivalent to the following two famous inequalities:

Theorem 4.1 (Mahler, Blaschke-Santaló)**.** *If K is a centrally symmetric convex body in the plane then*

$$8 \leq \lambda(K)\lambda^*(K^\circ) \leq \pi^2$$

with equality on the left if and only if K is a parallelogram and on the right if and only if K is an ellipse.

The inequality on the left is due to Mahler ([14]). Proving its generalization to higher dimensions is the famous *Mahler conjecture.* The two-dimensional version of the inequality on the right is due to Blaschke ([6]) and its extension to higher dimension is due to Santaló ([19]).

Form the previous theorem, we immediately obtain the following inequalities:

Corollary 4.1. *If $(\mathbb{R}^2, \|\cdot\|)$ is a Minkowski plane with unit disc D, then*

$$8/\pi \leq \mu_{ht}(D) \leq \pi$$

with equality on the left if and only if D is a parallelogram and on the right if and only if D is an ellipse.

The analogue inequalities for mass and mass* are given in the following two theorems:

Theorem 4.2. *If $(\mathbb{R}^2, \|\cdot\|)$ is a Minkowski plane with unit disc D, then*

$$2 \leq \mu_m(D) \leq \pi$$

with equality on the left if and only if D is a parallelogram and on the right if and only if D is an ellipse.

Theorem 4.3. *If $(\mathbb{R}^2, \|\cdot\|)$ is a Minkowski plane with unit disc D, then*

$$3 \leq \mu_{m*}(D) \leq 4$$

with equality on the left if and only if D is an affine regular hexagon and on the right if and only if D is a parallelogram.

These results will be proved in the next section, but two of these inequalities are really quite simple:

Exercise 4.1. Show that $2 \leq \mu_m(D)$ and that $\mu_{m*}(D) \leq 4$ for all unit discs with equality if and only if the unit disc is a parallelogram.

5. Area versus perimeter

The main theorem in this section relates the perimeter of the unit disc, D, of a Minkowski plane to the areas of D given by μ_m and μ_{m*}. The proof makes use of the simplifying assumption that ∂D is smooth. Since any convex body can be approximated by convex bodies with smooth boundaries and the quantities investigated are continuous in the Hausdorff topology (see [22], pp. 64), the proof below and an approximation argument yield the proof in the general case.

Theorem 5.1. *If* $(\mathbb{R}^2, \|.\|)$ *is a two dimensional normed space with unit disc* D *then*

$$2\mu_m(D) \leq \ell(\partial D) \leq 2\mu_{m*}(D).$$

Moreover, when ∂D *is a regular curve, we have equality on either side if and only if* ∂D *is a Radon curve.*

Proof. Let $\gamma : [0, \ell] \mapsto \mathbb{R}^2$ be a positively-oriented parameterization of ∂D by Minkowski arc-length as parameter (i.e., for every value of the parameter $\dot\gamma(s)$ is a unit vector and $(\gamma(s), \dot\gamma(s))$ is a positively oriented basis of $\mathbb{R}^2$. By Green's theorem

$$\lambda(D) = \frac{1}{2}\int_\gamma x\,dy - y\,dx = \frac{1}{2}\int_0^\ell \det(\gamma(s), \dot\gamma(s))\,ds.$$

For the inequality on the left we argue as follows: since $\dot\gamma(s)$ is a Minkowski unit vector, the parallelogram P_s with vertices $\gamma(s)$, $\dot\gamma(s)$, $\gamma(s)$, and $-\dot\gamma(s)$ is inscribed in D and hence has area no larger than that of a maximal inscribed parallelogram P_i. Moreover, $\det(\gamma(s), \dot\gamma(s)) = \lambda(P_s)/2$.

Thus we have

$$\lambda(D) = \frac{1}{4}\int_0^\ell \lambda(P_s)ds \leq \frac{1}{4}\int_0^\ell \lambda(P_i)ds = \lambda(P_i)\ell(\partial D)/4.$$

Since $\mu_m(D) = 2\lambda(D)/\lambda(P_i)$, we get the desired inequality.

To prove the inequality in the right notice that if $\Delta(s) := \det(\gamma(s), \dot\gamma(s))$ reaches a minimum at u, then $\gamma(u)$ is parallel to $\ddot\gamma(u)$. Indeed, $0 = \Delta^{'}(u) = \det(\gamma(u), \ddot\gamma(u))$. In other words, the tangent to ∂D at $\dot\gamma(u)$ is parallel to $\gamma(u)$. Thus the tangents to ∂D at $\pm\gamma(u)$ and $\pm\dot\gamma(u)$ form a circumscribing parallelogram to D. The area of this parallelogram is $4\det(\gamma(u), \dot\gamma(u))$ and cannot be less than that of a minimal circumscribing parallelogram, P_c.

Therefore, for all s we have $4\det(\gamma(s), \dot\gamma(s)) \geq \lambda(P_c)$. Thus,

$$8\lambda(D) \geq \int_0^\ell \lambda(P_c)\,ds = \ell(\partial D)\lambda(P_c),$$

from which we get $\ell(\partial D) \leq 2\mu_{m*}(D)$.

From the proofs of both inequalities we see that equality holds if and only if $\det(\gamma(s), \dot{\gamma}(s))$ is constant and hence, exercise 3.2 tells us that ∂D is a Radon curve. □

It is interesting to consider the case of equality in the previous theorem when the unit circle is not necessarily regular. Since every Radon curve can be approximated by smooth ones, we have that equality on both sides holds for general Radon curves. However, the equality $2\mu_{m*}(D) = \ell(\partial D)$ also holds for equiframed curves, as illustrated in the following exercise (*cf.,* [15]):

Exercise 5.1. Show that if the unit circle is a regular octagon, then both the perimeter and twice the mass* of the unit disc equal $16\tan(\pi/8)$.

We are now in a position to finish the proof of theorem 4.3.

Corollary 5.1. *If* $(\mathbb{R}^2, \|.\|)$ *is a two dimensional normed space with unit disc* D*, then* $\mu_{m*}(D) \geq 3$ *with equality if and only if* D *is a regular hexagon.*

Proof. By the previous theorem, $2\mu_{m*}(D) \geq \ell(\partial D)$ with equality if ∂D is a Radon curve, while Gołąb's theorem states that $\ell(\partial D) \geq 6$ with equality if and ∂D is a regular hexagon. Using both inequalities, together with the fact that the regular hexagon is a Radon curve, proves the corollary. □

To end this section we state without proof a result of Moustafaev and use it in conjunction with theorem 5.1 to finish the proof of theorem 4.2.

Theorem 5.2 (Moustafaev, [16])**.** *If* $(\mathbb{R}^2, \|.\|)$ *is a normed plane with unit disc* D *then*

$$2\mu_{ht}(D) \leq \ell(\partial D)$$

with equality if and only if D *is an ellipse.*

Moustafaev's proof requires more machinery than the proofs in this paper. It makes use of the solution of the isoperimetric problem in Minkowski planes and the Blaschke-Santaló inequality.

Corollary 5.2. *If* $(\mathbb{R}^2, \|.\|)$ *is a normed plane with unit disc* D *then*

$$\mu_{ht}(D) \leq \mu_{m*}(D) \quad \textit{and} \quad \mu_m(D) \leq \mu_b(D) = \pi$$

with equality of and only if D *is an ellipse.*

Proof. By theorem 5.1 and Moustafaev's result, we have that

$$2\mu_{ht}(D) \leq \ell(\partial D) \leq 2\mu_{m*}(D).$$

By exercise 3.6 — and using that μ_b, μ_{ht}, and μ_m, μ_{m*} are dual pairs of area definitions — we have that $\mu_m(D) \leq \mu_b(D) = \pi$. □

6. Properties of Radon curves

In this section we sharpen the bounds on the perimeter and area of the unit disc in the case where the unit circle is a Radon curve. We begin by putting together several of the results in the previous section.

Theorem 6.1. *If* $(\mathbb{R}^2, \|.\|)$ *is a normed plane with unit disc* D *and if* ∂D *is a Radon curve then*

$$(\ell(\partial D))^2 = 4\lambda(D)\lambda(D^\circ).$$

Proof. By exercise 3.4,

$$4\lambda(D)\lambda(D^\circ) = 4\mu_m(D)\mu_{m*}(D^\circ),$$

while theorem 5.1 tells us that $2\mu_m(D) = \ell(\partial D)$ and that $2\mu_{m*}(D^\circ) = \ell(\partial D^\circ)$. We now apply Schäffer's theorem, $\ell(\partial D^\circ) = \ell(\partial D)$, to obtain the desired result. □

As a corollary, we have the following result of Yaglom ([25]).

Corollary 6.1. *If* $(\mathbb{R}^2, \|.\|)$ *is a normed plane with unit disc* D *and if* ∂D *is a Radon curve then*

$$6 \leq \ell(\partial D) \leq 2\pi.$$

Proof. The first inequality follows from Gołąb's theorem. Equality is attained if an only if ∂D is an affine regular hexagon (which is a Radon curve). For the second inequality we have

$$(\ell(\partial D))^2 = 4\lambda(D)\lambda(D^\circ) \leq 4\pi^2$$

from the theorem above and the Blaschke-Santaló inequality. Here equality holds if and only if D is an ellipse. □

Corollary 6.2. *If* $(\mathbb{R}^2, \|.\|)$ *is a normed plane with unit disc* D *and if* ∂D *is a Radon curve then:*

(1) $3 \leq \mu_m(D) \leq \pi$;

(2) $3 \leq \mu_{m*}(D) \leq \pi$;

(3) $9/\pi \leq \mu_{ht}(D) \leq \pi$.

In all of these cases equality holds on the left if and only if D *is an affine regular hexagon and on the right if and only if* D *is an ellipse.*

Proof. The first two are evident from corollary 6.1 and the equality case of theorem 5.1. For the third note that from corollary 6.1

$$36 \leq (\ell(\partial D))^2 = 4\lambda(D)\lambda(D^\circ) \leq 4\pi^2.$$

If we now divide this equation by 4π and use the definition of $\mu_{ht}(D)$ we get the result. $\square$

7. Proof of Schäffer's Theorem

We end the paper by giving a "book proof" of Schäffer's theorem inspired by the symplectic proof of the girth conjecture given in [1].

Theorem. *If D is the unit disc for a Minkowski plane and if D° is the unit disc in the dual plane, then*

$$\ell(\partial D) = \ell(\partial D^\circ).$$

Proof. Consider the closed curve $\Gamma \subseteq \mathbb{R}^2 \times (\mathbb{R}^2)^*$ defined by

$$\Gamma := \{(q,p) : q \in \partial D\}$$

where, as in Lemma 4, we set $p := \mathcal{L}(\dot{q})$. Then

$$0 = \int_\Gamma d(p\cdot q) = \int_\Gamma p\cdot dq + q\cdot dp = \int_\Gamma p\cdot dq + \int_\Gamma q\cdot dp.$$

If we first parameterize ∂D by its own Minkowski arc length s then $\Gamma = \Gamma(s) = (q(s), \mathcal{L}(\dot{q}(s)))$ and so

$$\int_\Gamma p\cdot dq = \int_0^{\ell_1} \mathcal{L}(\dot{q}(s))\cdot \dot{q}(s)\, ds = \int_0^{\ell_1} ds = \ell_1$$

where $\ell_1 = \ell(\partial D)$.

On the other hand we can parameterize ∂D° by its Minkowski arc length t in which case, by proposition 3.1, $\Gamma = \Gamma(t) = (-\mathcal{L}(\dot{p}(t)), p(t))$. Then

$$\int_\Gamma q\cdot dp = -\int_0^{\ell_2} \mathcal{L}(\dot{p}(t))\cdot \dot{p}(t)\, dt = -\int_0^{\ell_2} ds = -\ell_2,$$

where $\ell_2 = \ell(\partial D^\circ)$.

In conclusion we get $0 = \ell(\partial D) - \ell(\partial D^\circ)$. $\square$

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