(1) (Definitions) Determine whether each of the following statements is True or False. (You do not have to explain.)

(a) If the augmented matrix of a system of linear equations has a row of zeroes in its reduced row echelon form, then the system has infinitely many solutions over \( \mathbb{C} \).

(b) If \( A \) and \( B \) are two \( n \times n \) invertible matrices, then \( ABA^{-1}B^{-1} = I \).

(c) If \( C \) is an orthogonal matrix, then \( \det(C) \) is equal to 1 or \(-1\).

(d) If \( A \) and \( B \) are two \( n \times n \) matrices such that \( \det(A) = \det(B) \) and \( \text{tr}(A) = \text{tr}(B) \), then \( A = B \).

(e) If \( A \) is a square matrix such that \( \det(A^{-1}) = \det(\text{adj}(A)) \), then \( \det(A) = 1 \).
(2) (Worksheet 3 & Sample Exam) Fill in the blanks. (You do not have to explain.)

(a) If $AB^2C$ is a $4 \times 9$ matrix and $BA^T$ is a $3 \times 4$ matrix, then the size of $A$ is ___________, the size of $B$ is ___________, and the size of $C$ is ___________.

(b) Under what conditions on the parameter $a$ is the matrix

$$
\begin{pmatrix}
 a & 8 & 0 & 0 \\
 2 & a & 0 & 0 \\
 -1 & 0 & 2 & a \\
 5 & a & -3 & 1 \\
\end{pmatrix}
$$

invertible over $\mathbb{R}$? _______________ (Hint: Block matrix.)

(c) If

$$
\begin{pmatrix}
 a_1 & a_2 & a_3 & a_4 \\
 b_1 & b_2 & b_3 & b_4 \\
 c_1 & c_2 & c_3 & c_4 \\
 d_1 & d_2 & d_3 & d_4 \\
\end{pmatrix}
$$

has determinant $4$, then

$$
\begin{vmatrix}
 a_1 & a_2 & a_3 & a_4 \\
 5b_1 & 5b_2 & 5b_3 & 5b_4 \\
 c_1 & c_2 & c_3 & c_4 \\
 2a_1 - d_1 & 2a_2 - d_2 & 2a_3 - d_3 & 2a_4 - d_4 \\
\end{vmatrix}
$$

= _______________.

(d) If $D$ is a $3 \times 4$ matrix over $\mathbb{R}$, and $\vec{b}$ is some vector in $\mathbb{R}^3$, which of the following scenario(s) can occur?

(i) The system $D\vec{x} = \vec{b}$ has no solutions.

(ii) The system $D\vec{x} = \vec{b}$ has exactly one solution.

(iii) The system $D\vec{x} = \vec{b}$ has infinitely many solutions.

ANSWER: ______________.
(3) (Worksheet 3) Fill in the blanks. You do not need to explain.

(a) The adjoint of the matrix 
\[
\begin{pmatrix}
1 & 2 & -1 \\
2 & 2 & 4 \\
1 & 3 & -3
\end{pmatrix}
\]
is

______________________________.

(b) The inverse of matrix from part (a), if it exists is ___________.
(4) Answer the following questions.
(a) Let $A$ be an $n \times n$ symmetric invertible matrix. Circle all true statements below.
   (i) $A^{-1}$ is also symmetric.
   (ii) $AA^T$ is also symmetric.
   (iii) $A - A^T$ is also symmetric.

(b) Let $A$ be a $3 \times 3$ matrix with $\det(A) = -4$ and $\text{tr}(A) = 16$, and let $B$ be a
    $3 \times 3$ matrix with $\det(B) = 9$ and $\text{tr}(B) = 5$. Evaluate each of the following
    expressions, or state if there is not enough information to evaluate.
    \[
    \det(2B^{-1}A^{-1}B^T) = \\
    \text{tr}(5I_3 + 2B) = \\
    \det(\det(A)A) = 
    \]
(5) (Worksheet 1) Consider the linear system

\[
\begin{align*}
    x + y + 2z &= 0 \\
    -2x - y - 7z &= 2 \\
    3x + (k + 5)y + k^2 z &= (k + 2)^2
\end{align*}
\]

in unknowns \( x \), \( y \), and \( z \) and where \( k \) is a complex parameter.

(a) For what value(s) of \( k \) does the system have infinitely many solutions over the complex numbers?

(b) For what value(s) of \( k \) is the system inconsistent?

(c) For what value(s) of \( k \) does the system have exactly one solution over the complex numbers?
(6) (Worksheet 2) Short proofs. Carefully show your explanation.

(a) Show that if $M$ is an $n \times n$ orthogonal matrix, then $\det(M) = \pm 1$.

(b) Show that if $C$ is a $2 \times 2$ matrix, then $\text{adj}(\text{adj}(C)) = C$. 
(7) (Worksheet 1) During one month, a small business used 600 units of electricity, 300 units of gas and 200 units of water for a total cost of $380. The next month 500 units of electricity, 400 units of gas and 150 units of water were used for a total cost of $345. The third month the consumption was 450 units of electricity, 300 units of gas and 100 units of water for a total cost of $295. Set up the system of linear equations that you could use to find the cost per unit of electricity, the cost per unit of gas, and the cost per unit of water and solve the system using the Gaussian Algorithm. Show all of your work.
(8) Answer the following questions.

(a) Show that the matrix equation $AB - BA = I$ has no solutions in the $n \times n$ matrix field with real entries. Write clearly your entire explanation.

(b) Let $A$ be an $n \times n$ matrix such that

$$(A - I_n)(A - 2I_n)(A - 3I_n) = 0,$$

where $0$ denotes the $n \times n$ zero matrix. Show that $A$ has an inverse, and find its inverse.
(9) Determine whether each of the following statements is True or False. You need not explain.

(a) $S_1 = \{ (x, y, z) \in \mathbb{R}^3 : x = 1 \}$, with the operations addition and scalar multiplication in the usual sense, constitutes a real vector space.

(b) $S_2 = \{ (x, y, z) \in \mathbb{R}^3 \}$, with addition in the usual sense, and scalar multiplication defined as $k(u) = (kx, ky, 0)$ for every $k \in \mathbb{R}, u \in \mathbb{R}^3$, constitutes a real vector space.

(c) $S_3 = \{ p(t) = at^2 + bt + c \in \mathcal{P}_2 : p'(0) = 0 \}$ with addition and scalar multiplication in the usual sense, constitutes a real vector space.

(d) The set of all real numbers which are even, with addition and scalar multiplication in the usual sense, constitutes a real vector space.

(e) The set of all rational numbers is closed under usual addition.

(f) A vector space may contain exactly 2 elements if one of the elements is the zero vector.

(10) Which of the following sets form subspaces?

(a) Let $V$ be the vector space of all real-valued functions defined on $\mathbb{R}$, i.e., $f : \mathbb{R} \rightarrow \mathbb{R}$. Determine whether $W$ is a subspace of $V$.

(i) $W_1 = \{ f \in V : f(0) = 0 \text{ and } f(1) = 1 \}$

(ii) all odd functions, i.e., $W_2 = \{ f \in V : f(-x) = -f(x) \}$

(iii) all polynomials (no restriction on degree)

(b) All invertible $n \times n$ matrices

(c) $\{(x, y) \in \mathbb{R}^2 : x - y = 1 \}$

(d) All polynomials of degree at most 3, with integers as coefficients.

(e) All polynomials in $\mathcal{P}_n$ such that $p(0) = 0$.

(f) $\{ at^2 + bt + c \in \mathcal{P}_2 : a + b = c \}$

(g) all symmetric $5 \times 5$ matrices $\in \mathcal{M}_{5\times5}$

(h) all singular matrices

(i) all matrices

\[
\begin{pmatrix}
  a & b & c \\
  d & e & f
\end{pmatrix} \in \mathcal{M}_{2\times3}
\]

such that $a = -2d$ and $f = 2e + c$, for $a,b,c,d,e,f \in \mathbb{R}$