Appendix B

The Laplace Expansion For Determinant

B.1 First Minors and Cofactors; Row and Column Expansions

To each element $a_{ij}$ in the determinant $|A| = |a_{ij}|_n$, there is associated a subdeterminant of order $(n - 1)$ which is obtained from $|A|$ by deleting row $i$ and column $j$. This subdeterminant is known as a first minor of $|A|$ and is denoted by $M_{ij} = D(A(i|j))$. The first cofactor $(C_A)_{ij}$ is then defined as a signed first minor:

$$(C_A)_{ij} = (-1)^{i+j} M_{ij} = (-1)^{i+j} D(A(i|j)).$$

(B.1.1)

It is customary to omit the adjective first and to refer simply to minors and cofactors and it is convenient to regard $M_{ij}$ and $(C_A)_{ij}$ as quantities which belong to $a_{ij}$ in order to give meaning to the phrase “an element and its cofactor.”

The expansion of $|A|$ by elements from row $i$ and their cofactors is

$$|A| = D(A) = \sum_{j=1}^{n} a_{ij}(C_A)_{ij} = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} D(A(i|j)), \quad 1 \leq i \leq n. \quad (B.1.2)$$

The expansion of $|A|$ by elements from column $j$ and their cofactors is

$$|A| = D(A) = \sum_{i=1}^{n} a_{ij}(C_A)_{ij} = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} D(A(i|j)), \quad 1 \leq j \leq n. \quad (B.1.3)$$

Since $(C_A)_{ij}$ belongs to but is independent of $a_{ij}$, another way to define $(C_A)_{ij}$ is

$$(C_A)_{ij} = \frac{\partial |A|}{\partial a_{ij}}. \quad (B.1.4)$$
\section*{B.2 Alien Cofactors; The Sum Formula}

The theorem on alien cofactors states that
\begin{equation}
\sum_{j=1}^{n} a_{ij}(C.A)_{kj} = 0, \quad 1 \leq i \leq n, \quad 1 \leq k \leq n, \quad k \neq i. \tag{B.2.1}
\end{equation}

The elements come from row $i$ of $|A|$, but the cofactors belong to the elements in row $k$ and are said to be alien to the elements. The identity is merely an expansion by elements from row $k$ of the determinant in which row $k = row$ and which therefore zero.

The identity can be combined with the expansion formula for $A$ with the aid of the Kronecker delta function $\delta_{ik}$ to form a single identity which may be called the sum formula for elements and cofactors:
\begin{equation}
\sum_{j=1}^{n} a_{ij}(C.A)_{kj} = \delta_{ik}|A|, \quad 1 \leq i \leq n, 1 \leq k \leq n. \tag{B.2.2}
\end{equation}

It follows that
\begin{equation}
\sum_{j=1}^{n} (C.A)_{ij}C_j = \begin{bmatrix}
0 \\
\vdots \\
0 \\
D(A) \\
\vdots \\
0
\end{bmatrix}, \quad 1 \leq i \leq n, \tag{B.2.3}
\end{equation}

where $C_j$ is column $j$ of the matrix $A$, and the element $D(A)$ is in the row $i$ of the column vector and all the other elements are zero. If $D(A) = 0$, then
\begin{equation}
\sum_{j=1}^{n} (C.A)_{ij}C_j = 0, \quad 1 \leq i \leq n, \tag{B.2.4}
\end{equation}

that is, the columns are linearly dependent. Conversely, if the columns are linearly dependent, then $D(A) = 0$. 

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B.3 Cramer’s Formula

The set of equations

\[
\sum_{j=1}^{n} a_{ij} x_j = b_i, \quad 1 \leq i \leq n, \quad (B.3.1)
\]

can be expressed in column vector notation as follows:

\[
\sum_{j=1}^{n} C_j x_j = B, \quad (B.3.2)
\]

where

\[
B = \begin{bmatrix}
    b_1 \\
    b_2 \\
    b_3 \\
    \vdots \\
    b_n
\end{bmatrix}.
\]

So

\[
\begin{vmatrix}
    C_1 & \cdots & C_{j-1} & B & C_{j+1} & \cdots & C_n
\end{vmatrix}
= \begin{vmatrix}
    C_1 & \cdots & C_{j-1} & \sum_{j=1}^{n} C_j x_j & C_{j+1} & \cdots & C_n
\end{vmatrix}
\]

\[
= \begin{vmatrix}
    C_1 & \cdots & C_{j-1} & x_1 C_1 & C_{j+1} & \cdots & C_n
\end{vmatrix}
+ \cdots
\]

\[
+ \begin{vmatrix}
    C_1 & \cdots & C_{j-1} & x_{j-1} C_{j-1} & C_{j+1} & \cdots & C_n
\end{vmatrix}
+ \begin{vmatrix}
    C_1 & \cdots & C_{j-1} & x_j C_j & C_{j+1} & \cdots & C_n
\end{vmatrix}
+ \begin{vmatrix}
    C_1 & \cdots & C_{j-1} & x_{j+1} C_{j+1} & C_{j+1} & \cdots & C_n
\end{vmatrix}
+ \cdots
\]

\[
+ \begin{vmatrix}
    C_1 & \cdots & C_{j-1} & x_n C_n & C_{j+1} & \cdots & C_n
\end{vmatrix}
= x_j \begin{vmatrix}
    C_1 & \cdots & C_{j-1} & C_j & C_{j+1} & \cdots & C_n
\end{vmatrix}
= x_j |A|. \quad (B.3.3)
\]
If $|A| = |a_{ij}| \neq 0$, then the unique solution of the equations can also be expressed in column vector notation.

$$x_j = \frac{1}{|A|} \begin{vmatrix} C_1 & \cdots & C_{j-1} & B & C_{j+1} & \cdots & C_n \end{vmatrix}$$

$$= \frac{1}{|A|} \sum_{i=1}^{n} b_i (\mathcal{E}A)_{ij}.$$

(B.3.4)