Directions: You have 90 minutes to answer the following questions. You must show all your work as neatly and clearly as possible and indicate the final answer clearly. You may use a calculator, but you must show your work for integrals and derivatives. There are formulas on the last page of the exam which you may detach.

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(1) (20 points) Consider the initial value problem

\[ P' = e^{P - 2t}, \quad P(1) = 1 \]

(a) Find an **explicit** formula for \( P(t) \).

(b) What is the interval of existence for this solution?
(2) (20 points) In this problem we consider the autonomous ODE of the form \( y' = f(y) \), where \( f(y) = -y^3 - 3y^2 + 4y + 12 \).

(a) Sketch a graph of \( f(y) \) and use it to develop a **phase line** for the ODE which exhibits the **equilibrium points** and classifies them as being (asymptotically) **stable** or **unstable**.

(b) If \( y(-2) = 0 \), then \( \lim_{t \to \infty} y(t) = \underline{\underline{\text{ }}} \).

(c) If \( y(0) = -3 \), then \( \lim_{t \to \infty} y(t) = \underline{\underline{\text{ }}} \).

(d) If \( y(0) = -2 \), then \( y(-3) = \underline{\underline{\text{ }}} \).
(3) (20 points) Solve the initial value problem for the second order ODE
\[ y'' - 2y' = 1 + e^x + e^{2x}, \quad y(0) = 1, \quad y'(0) = -1 \]
(4) (20 points) Consider the homogeneous, second order linear ODE
\[ x^2y'' + 3xy' + y = 0, \text{ for } x > 0 \]

(a) Verify that \( y_1 = \frac{1}{x} \) is a solution.

(b) Use the substitution \( y = vy_1 \) to show that \( y_2 = \frac{\ln(x)}{x} \) is another solution and then use the Wronskian to verify that \( y_1 \) and \( y_2 \) are linearly independent.

(c) Solve the IVP
\[ x^2y'' + 3xy' + y = 0, \quad y(1) = e, \quad y'(1) = \frac{1}{e} \]
(5) (20 points) A bacteria population $P(t)$ grows according to the logistic equation

$$P' = r_0 (1 - \frac{P}{K}) P$$

Suppose the initial population is 20 percent of the carrying capacity, and the population doubles after one hour.

(a) What is the natural reproductive rate for this population?
(b) At what time does the population reach 80 percent of its carrying capacity?


**Formula Sheet**

(1) **Integration By Parts:** \[ \int u(x)v'(x) \, dx = u(x)v(x) - \int u'(x)v(x) \, dx \]

(2) **Partial Fractions Integral:** If \( c \neq d \) then

\[
\int \frac{ax + b}{(x - c)(x - d)} \, dx = \frac{1}{c - d} \left( (ac + b) \ln |x - c| - (ad + b) \ln |x - d| \right) + K
\]

(3) **The Logistic Equation:** \( P' = r_0(1 - P/K)P \) has the implicit general solution

\[
\frac{P}{K - P} = \frac{P_0}{K - P_0} e^{r_0 t}
\]

(4) **Variation of Parameters:** If \( y_1 \) and \( y_2 \) are linearly independent solutions of the equation \( y'' + p(t)y' + q(t)y = 0 \), then \( y_p = v_1 y_1 + v_2 y_2 \) is a particular solution of the equation \( y'' + p(t)y' + q(t)y = f(t) \), where \( v_1 \) and \( v_2 \) satisfy the VOP equations

\[
\begin{align*}
v_1' y_1 + v_2' y_2 &= 0 \\
v_1' y_1' + v_2' y_2' &= f(t).
\end{align*}
\]