Directions: You have two hours to answer the following questions. You must show all your work as neatly and clearly as possible and indicate the final answer clearly. You may use a calculator.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Possible</th>
<th>Points</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>25</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>15</td>
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<tr>
<td>3</td>
<td>20</td>
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<tr>
<td>4</td>
<td>15</td>
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<tr>
<td>5</td>
<td>15</td>
<td></td>
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<tr>
<td>6</td>
<td>10</td>
<td></td>
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<tr>
<td>Total</td>
<td>100</td>
<td></td>
</tr>
</tbody>
</table>
(1) (25 points) Phone calls arrive at an exchange at the rate of eight calls per minute, and the pattern they arrive can be modeled by a Poisson distribution.

(a) Find the probability that at most 4 calls arrive in a 1-minute period. (Give your answer accurate to 2 decimal places.)

(b) Find the probability that the waiting time for the first phone call is more than 20 seconds.

(c) Estimate the probability that there are more than 500 calls in a 1-hour period.

(d) If we are observing the numbers of calls in each minute in a 1-hour period, let $Y$ be the number of minutes where there were less than or equal to 4 calls coming in, find $P(Y \geq 2)$. 
(2) (15 points)
(a) Suppose that the SAT scores of all the applicants to a certain college follow a normal distribution. What percentage of applicants have SAT scores that are at least 1 standard deviation higher than the average SAT scores of all applicants?

(b) Suppose there are 2000 applicants, and the SAT scores follow approximately a normal distribution with a mean of 1350 and a standard deviation of 100. If the college requires SAT score of at least 1200, how many of these students will be rejected on this basis regardless of their other qualifications?
(3) (20 points) Let $X$ be a uniform distribution over the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

(a) Write the p.d.f of $X$.

(b) Define $Y = \tan X$. Find the p.d.f. of $Y$. (This is a widely used random variable, it is called the Cauchy random variable.)

(c) For the Cauchy random variable $Y$, find $P(Y \geq 10)$.

(d) For the Cauchy random variable $Y$, find $E(Y)$ if it exist. If not, explain why.
(4) (15 points) Many people believe that the daily change in price of a company’s stock in the stock market is a random variable with mean 0 and variance $\sigma^2$ during middle 2 months between 2 earning report dates when there is not much news released. That is, if $Y_n$ represents the price of the stock on the $n$th day of that period, then

$$Y_n = Y_{n-1} + X_n, \quad 1 \leq n \leq 60,$$

where $X_1, X_2, \ldots, X_{60}$ are independent and identically distributed random variables with mean 0 and variance $\sigma^2$. Suppose that the stock’s price at the beginning of the 2-month period is 100. If $\sigma = 2$, what can you say about the probability that the stock’s closing price will exceed 110 on the 30th day?
(5) (15 points) A club basketball team will play a 74-game season. Forty-four of these games are against class A teams and 30 are against class B teams. Suppose that the team will win each game against a class A team with probability 0.4, and will win each game against a class B team with probability 0.7. Assume also that the results of the different games are independent. Approximate the probability that the team will win more games against class A teams than it does against class B teams.
(6) (10 points)

(a) Use the moment generating function to derive the distribution of \( Y = X_1 + X_2 \), where \( X_1 \) and \( X_2 \) are independent samples from an exponential distribution with \( \theta = 10 \).

(b) Prove the following analytical identity using a probability argument:

\[
\sum_{i=k}^{n} \binom{n}{i} x^i (1-x)^{n-i} = \frac{n!}{(k-1)!(n-k)!} \int_{0}^{x} y^{k-1} (1-y)^{n-k} \, dy, \quad 0 \leq x \leq 1.
\]
Some definitions and formulas you might find useful

(1) For a random sample \(x_1, x_2, \ldots, x_n\), the sample mean and the sample variance are defined as

\[
\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i \quad \text{and} \quad s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2.
\]

(2) If \(0 < p < 1\), the \((100p)\)th sample percentile of \(x_1, x_2, \ldots, x_n\) has approximately \(np\) sample observations less than it and also \(n(1-p)\) sample observations greater than it. If \((n+1)p\) is an integer \(r\) plus some proper fraction \(a/b\), one way to define the \((100p)\)th sample percentile \(\tilde{\pi}_p\) is

\[
\tilde{\pi}_p = x_{(r)} + \frac{a}{b} (x_{(r+1)} - x_{(r)}) = (1 - \frac{a}{b}) x_{(r)} + \frac{a}{b} x_{(r+1)},
\]

where \(x_{(r)}\) and \(x_{(r+1)}\) are the \(r\)th and \((r+1)\)th order statistics.

(3) \(P(n, r) = \frac{n!}{(n-r)!}; \quad C(n, r) = \binom{n}{r} = \frac{n!}{r!(n-r)!}\).

(4) The conditional probability of \(A\), given \(B\), is \(P(A|B) = \frac{P(A \cap B)}{P(B)}\), if \(P(B) > 0\).

(5) The cumulative distribution function \(F(x)\) of a continuous random variable \(X\) with p.d.f. \(f(x)\) is \(F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x) \, dx\). Replace the integral with a sum for discrete case.

(6) Assume that \(X\) is a continuous random variable with p.d.f. \(f(x)\). Then the expectation and the variance \(X\) are defined as,

\[
E(X) = \int_{-\infty}^{\infty} xf(x) \, dx; \quad \text{and,}
\]

\[
Var(X) = E[(X - E(X))^2] = E(X^2) - (E(X))^2.
\]

The moment generating function of \(X\), if it exists, is defined as

\[
M(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f(x) \, dx \quad -h < t < h.
\]

If \(X\) is a discrete random variable, just replace all the integrals with proper summations.

(7) The \(r\)th moment of a random variable \(X\) is defined as \(\mu_r = E(X^r)\).

(8) If \(X\) is \(N(\mu, \sigma^2)\), then \(Z = \frac{X - \mu}{\sigma}\) is \(N(0, 1)\).

(9) Suppose that \(X\) is a continuous random variable with p.d.f. \(f(x)\) with support \(S_x\). Let \(Y = u(X)\) define a one-to-one correspondence between the values of \(X\) and \(Y\).
so that the equation \( x = w(y) \) is the inverse function of \( y = u(x) \). Then the p.d.f. of \( Y \) is

\[
g(y) = f'[w(y)]|w'(y)|, \quad y \in S_y,
\]

where \( S_y \) is the support of \( Y \) found by mapping the support of \( X, S_x, \) onto it.

(10) Let \( X \) and \( Y \) have the joint probability mass function \( f(x,y) \) with space \( S \). The probability mass function of \( X \) alone, which is called the marginal probability mass function of \( X \), is defined by

\[
f_1(x) = P(X = x) = \sum_y f(x,y), \quad x \in S_1,
\]

where the summation is taken over all possible \( y \) values for each given \( x \) in the \( x \) space \( S_1 \). Similarly, the probability mass function of \( Y \) alone, which is called the marginal probability mass function of \( Y \), is defined by

\[
f_2(y) = P(Y = y) = \sum_x f(x,y), \quad y \in S_2,
\]

where the summation is taken over all possible \( x \) values for each given \( y \) in the \( y \) space \( S_2 \).

The random variables \( X \) and \( Y \) are independent if and only if

\[
f(x,y) = f_1(x)f_2(y), \quad x \in S_1, \quad y \in S_2;
\]

otherwise, \( X \) and \( Y \) are called dependent.

Similarly, you can define the marginal p.d.f.’s for continuous random variables, and the independence of continuous random variables.

(11) For random variables \( X \) and \( Y \), the covariance of \( X \) and \( Y \) is defined as

\[
Cov(X,Y) = \sigma_{XY} = E[(X - \mu_X)(Y - \mu_Y)] = E(XY) - E(X)E(Y).
\]

If \( X \) and \( Y \) both have positive variances, the correlation coefficient is defined as

\[
\rho = \frac{Cov(X,Y)}{\sigma_X \sigma_Y} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}.
\]

(12) Let \( X_1, X_2, \ldots, X_n \) be \( n \) independent random observations from a continuous distribution with distribution function \( F(x) \) and p.d.f. \( f(x) \), where \( 0 < F(x) < 1 \) for \( a < x < b \) and \( F(a) = 0, F(b) = 1. \) (It’s possible that \( a = -\infty \) and/or \( b = \infty \).) Let

\[
X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(n)}
\]

be the order statistics of \( X_1, X_2, \ldots, X_n \). Then the p.d.f of the \( r \)th order statistics \( X_{(r)} \) is

\[
g_r(x) = \frac{n!}{(r-1)!(n-r)!}[F(x)]^{r-1}[1 - F(x)]^{n-r} f(x), \quad a < x < b.
\]
(13) If \(X_1, X_2, \ldots, X_n\) are independent random variables with respective moment-generating functions \(M_{X_i}(t), i = 1, 2, \ldots, n\), then the moment-generating function of \(Y = \sum_{i=1}^{n} a_i X_i\) is
\[
M_Y(t) = \prod_{i=1}^{n} M_{X_i}(a_i t).
\]

(14) If \(X_1, X_2, \ldots, X_n\) are observations of a random sample of size \(n\) from the normal distribution \(N(\mu, \sigma^2)\), then \(\frac{(n-1)S^2}{\sigma^2}\) follows a Chi-square distribution with \(n - 1\) degrees of freedom, where \(S^2\) is the sample variance.

(15) (Central Limit Theorem) If \(\bar{X}\) is the mean of a random sample \(X_1, X_2, \ldots, X_n\) of size \(n\) from a distribution with a finite mean \(\mu\) and a finite positive variance \(\sigma^2\), then the distribution of
\[
W = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} = \frac{\sum_{i=1}^{n} X_i - n\mu}{\sqrt{n} \sigma}
\]
is \(N(0, 1)\) in the limit as \(n \to \infty\).
A table for some well-known distributions. (Note: In the following table \( q = 1 - p \).)

<table>
<thead>
<tr>
<th>Name</th>
<th>p.d.f ( f(x) )</th>
<th>m.g.f ( M(t) )</th>
<th>mean ( \mu )</th>
<th>variance ( \sigma^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bernoulli(( p ))</td>
<td>( f(x) = p^x q^{1-x}, \ x = 1, 2 )</td>
<td>( M(t) = q + pe^t )</td>
<td>( p )</td>
<td>( pq )</td>
</tr>
<tr>
<td>Binomial(( n, p ))</td>
<td>( f(x) = \binom{n}{x} p^x q^{n-x} ) ( x = 0, 1, 2, \ldots, n )</td>
<td>( M(t) = (q + pe^t)^n )</td>
<td>( np )</td>
<td>( npq )</td>
</tr>
<tr>
<td>Geometric(( p ))</td>
<td>( f(x) = q^{x-1} p ) ( x = 1, 2, \ldots )</td>
<td>( M(t) = \frac{pe^t}{1 - qe^t} )</td>
<td>( \frac{1}{p} )</td>
<td>( \frac{q}{p^2} )</td>
</tr>
<tr>
<td>Negative Binomial (( r, p ))</td>
<td>( f(x) = (\frac{x-1}{r-1}) p^r q^{x-r} ) ( x = r, r+1, r+2, \ldots )</td>
<td>( M(t) = \frac{(pe^t)^r}{(1 - qe^t)^r} )</td>
<td>( \frac{r}{p} )</td>
<td>( \frac{rq}{p^2} )</td>
</tr>
<tr>
<td>Poisson(( \lambda ))</td>
<td>( f(x) = \frac{e^{-\lambda} \lambda^x}{x!} ) ( x = 0, 1, 2, \ldots )</td>
<td>( M(t) = e^{\lambda(e^t-1)} )</td>
<td>( \lambda )</td>
<td>( \lambda )</td>
</tr>
<tr>
<td>Exponential(( \theta ))</td>
<td>( f(x) = \frac{1}{\theta} e^{-x/\theta} ) ( 0 \leq x &lt; \infty )</td>
<td>( M(t) = \frac{1}{1 - \theta t} )</td>
<td>( \theta )</td>
<td>( \theta^2 )</td>
</tr>
<tr>
<td>Gamma (( \alpha, \theta ))</td>
<td>( f(x) = \frac{1}{\Gamma(\alpha) \theta^\alpha} x^{\alpha-1} e^{-x/\theta} ) ( 0 \leq x &lt; \infty )</td>
<td>( M(t) = \frac{1}{(1 - \theta t)^\alpha} )</td>
<td>( \alpha \theta )</td>
<td>( \alpha \theta^2 )</td>
</tr>
<tr>
<td>Chi-Square(( r ))</td>
<td>( f(x) = \frac{1}{\Gamma(\frac{r}{2}) 2^\frac{r}{2}} x^{r/2-1} e^{-x/2} ) ( 0 \leq x &lt; \infty )</td>
<td>( M(t) = \frac{1}{(1 - 2t)^{r/2}} )</td>
<td>( r )</td>
<td>( 2r )</td>
</tr>
<tr>
<td>Normal ( N(\mu, \sigma^2) )</td>
<td>( f(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)} ) ( -\infty &lt; x &lt; \infty )</td>
<td>( M(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}} )</td>
<td>( \mu )</td>
<td>( \sigma^2 )</td>
</tr>
</tbody>
</table>
Differentiation formulas

<table>
<thead>
<tr>
<th>$\frac{d}{dx}(x^n)$</th>
<th>$d(e^x) = e^x$</th>
<th>$d(a^x) = (\ln a)a^x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$nx^{n-1}$</td>
<td>$\cos x$</td>
<td>$\cos x$</td>
</tr>
<tr>
<td>$\frac{d}{dx}(\ln</td>
<td>x</td>
<td>) = \frac{1}{x}$</td>
</tr>
<tr>
<td>$\sec^2 x$</td>
<td>$\sec x$</td>
<td>$\sec x$</td>
</tr>
<tr>
<td>$\sec x \tan x$</td>
<td>$\sec x$</td>
<td>$\sec x$</td>
</tr>
<tr>
<td>$\frac{d}{dx}(\arcsin(x)) = \frac{1}{\sqrt{1-x^2}}$</td>
<td>$\frac{d}{dx}(\arccos(x)) = -\frac{1}{\sqrt{1-x^2}}$</td>
<td>$\frac{d}{dx}(\arctan(x)) = \frac{1}{1+x^2}$</td>
</tr>
<tr>
<td>$\frac{d}{dx}(\arccos(x)) = -\frac{1}{\sqrt{1-x^2}}$</td>
<td>$\frac{d}{dx}(\csc(x)) = -\csc x \cot x$</td>
<td>$\frac{d}{dx}(\csc(x)) = -\csc x \cot x$</td>
</tr>
<tr>
<td>$\frac{d}{dx}(\cot(x)) = -\csc^2 x$</td>
<td>$\frac{d}{dx}(\csc(x)) = -\csc x \cot x$</td>
<td>$\frac{d}{dx}(\cot(x)) = -\csc^2 x$</td>
</tr>
</tbody>
</table>

Integration by Parts:

$$\int u dv = uv - \int v du$$

or

$$\int u dv = uv - \int v du$$

The $n$th degree Taylor Polynomial of $f(x)$ centered at $x = a$:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Taylor series of $f(x)$ centered at $x = a$:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

Taylor Series of important functions:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots, \quad \cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \quad \text{for} \quad -1 < x < 1$$

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad \text{for} \quad -1 < x < 1$$

$$(1 + x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots \quad \text{for} \quad -1 < x < 1$$

Finite Geometric Series:

$$a + ax + ax^2 + \cdots + ax^{n-1} = \frac{a(1 - x^n)}{1 - x}$$

Infinite Geometric Series:

$$a + ax + ax^2 + \cdots = \frac{a}{1 - x} \quad \text{for} \quad |x| < 1$$