(1) **Tautologies:** Prove that the following compound propositions are tautologies (that is, they are always true no matter what the truth values of the propositions that occur in them.

(1.b) \[ (p \rightarrow q) \land (q \rightarrow r) \rightarrow (p \rightarrow r) \]

By showing they are logically equivalent to T using the properties in Tables 5 and 6 (p.17-18) (See example 6 on p.18)

**Solution:**
\[ [(p \rightarrow q) \land (q \rightarrow r)] \rightarrow (p \rightarrow r) \]
\[ \iff [(\neg p \lor q) \land (\neg q \lor r)] \rightarrow (\neg p \lor r) \]
\[ \iff \neg[(\neg p \lor q) \land (\neg q \lor r)] \lor (\neg p \lor r) \]
\[ \iff [\neg(\neg p \lor q) \lor \neg(\neg q \lor r)] \lor (\neg p \lor r) \]
\[ \iff [(p \land \neg q) \lor (q \land \neg r)] \lor (\neg p \lor r) \]
\[ \iff [(p \land \neg q) \lor \neg p] \lor [(q \land \neg r) \lor r] \]
\[ \iff [(p \lor \neg p) \land (\neg q \lor \neg p)] \lor [(q \lor r) \land (r \lor \neg r)] \]
\[ \iff [T \land (\neg q \lor \neg p)] \lor [(q \lor r) \lor T] \]
\[ \iff (\neg q \lor \neg p) \lor (q \lor r) \]
\[ \iff (\neg q \lor q) \lor (\neg p \lor r) \]
\[ \iff T \lor (\neg p \lor r) \]
\[ \iff T \]

Make sure you understand which properties are being used at each step.

(2) Use logical reasoning (and not truth tables) to explain why
\[ (p \rightarrow q) \land (q \rightarrow r) \land (r \rightarrow p) \iff p \leftrightarrow q \leftrightarrow r \]

**Solution:**
To show that two statements P and Q are logically equivalent, you must show that they have the same truth values. It is NOT enough to show that if P is true, then Q is true - consider the statements P: \( x = 2 \) and Q: \( x^2 = 4 \), it is true that \( P \Rightarrow Q \) but P and Q are not logically equivalent because Q is true when \( x = -2 \) whereas P is false.

By definition \( p \leftrightarrow q \leftrightarrow r \) is true when \( p, q \) and \( r \) have the same truth values, and false otherwise. If \( p, q \), and \( r \) have the same truth values then \( (p \rightarrow q) \land (q \rightarrow r) \land (r \rightarrow p) \) is true because \( F \rightarrow F \) and \( T \rightarrow T \) are true. If \( p, q \), and \( r \) have different truth values then at least one is true and at least one is false, and since \( T \rightarrow F \) is false so is \( (p \rightarrow q) \land (q \rightarrow r) \land (r \rightarrow p) \).
(3) Is “This statement is false” a proposition? Explain.

**Solution:** The statement referred to is “This statement is false”. If the statement is true then the statement is false, and if the statement is false then the statement is true. Propositions are true or false but not both - so this statement is not a proposition.

Mathematical statements can be ambiguous when they refer to themselves, so you should avoid making/using such statements.

(4) **Knights and Knaves:** There are two kinds of inhabitant on an island: Knights, who always tell the truth, and Knaves, who always lie. There is no way to distinguish between knights and knaves visually. You are on the island and have the following encounters with the native inhabitants.

(a) You meet two people, A and B. A says “At least one of us is a knave” and B says nothing. Can you determine what A and B are, and if not, can you draw any conclusions?

**Solution:** If A is a knave then his statement is false, so there are no knaves - but this makes no sense because A is a knave. Therefore A is a knight, and since his statement is true, B is a knave.

(b) You then meet two more people, C and D. C says “We are both knights” and D says “C is a knave”. Can you determine what C and D are, and if not, can you draw any conclusions?

**Solution:** If C is a knight, then his statement is true so both C and D are knights, but then D must be telling the truth and so C is a knave which makes no sense, because we assumed C was a knight. Therefore C’s statement is false and he is a knave, and so D’s statement is true and he is a knight.

(c) You meet another two people who are guarding two doors, one is a knight and one is a knave (but you don’t know which is which). If you go through one door, you never have to take another math class again but if you go through the other door you are doomed to take math classes for another five years. You may ask ONLY one question and to ONLY one of the guards. What question should you ask to ensure that you choose the correct door, and never have to take another math class?

**Solution:** The point is that a knight tells the truth about a knave’s lies, and a knave lies about a knight telling the truth - so you must ask one a question about the other. Point at a door and ask either person

”If I ask the other person whether the door I am pointing to is the correct one, what will he say?”

If you are pointing to the correct door, the knight will say “no” because he knows the knave will lie, whereas the knave will say “no” because he lies about the knight telling the truth. If you are pointing to the wrong door then the knight will say “yes”, and the knave will say “yes”.
So if the answer is “no” you should take the door you are pointing to, and if the answer is “yes” you should take the other door.

(d) You meet one person. You may ask ONE question to determine whether the person is a knight or knave, provided it is not a question you already know the answer to (for example, “Is my hair brown?” or “Is 1+1=2?”). What question would you ask?

**Solution:** Ask the question “If I ask you whether you are knight, would you say yes?”. A knight will obviously say “yes”. If you ask a knave “are you a knight?” he would lie and say “yes”, but if you ask him the original question he would have to lie about his lie and say “no”. (If this doesn’t make sense, imagine that the knave thinks to himself “I would answer yes to that question” so now I have to lie about that answer and say “no”). The whole point is to ask a question within a question, to force the knave to lie about a lie and hence tell the truth.