(1) **Divisibility by 7:** Let \( m = d_n d_{n-1} \ldots d_3 d_2 d_1 d_0 \) be an \( n \)-digit integer. (So for example 1287 has \( d_0 = 7, d_1 = 8, d_2 = 2, d_4 = 1 \). Recall that in class we showed that \( m \) is divisible by 11 if and only if \( \sum_{i=0}^{n} (-1)^i d_i \) is divisible by 11.

(a) Find a similar (though slightly more complicated) formula involving the digits of \( m \) which determines whether \( m \) is divisible by 7. (Hint: We worked out \( 10^i \mod 7 \) in class.)

(b) Use the ideas from the first part to determine the remainder when 1462783394 is divided by 7. No credit for calculator answers.

(2) Prove that the set of bitstrings of infinite length is uncountable. (Hint: Assume the set is countable, list the elements and then show that there is always a bitstring which is not on the list to obtain a contradiction).

(3) Solve for \( x \). Your answer should have the form \( x \equiv a \mod m \) where \( 0 \leq a < m \).

(a) \( 30x \equiv 1 \mod 49 \).
(b) \( 2x \equiv 73 \mod 99 \).
(c) \( 9x \equiv 3 \mod 64 \).

(4) Prove the following results using mathematical induction. Note that we have already seen some of these results in previous worksheets/class but proved them using different methods, you must use induction to receive any credit.

(a) \( \sum_{i=1}^{n} (2i - 1) = n^2 \)

(b) \( \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \)

(c) \( \sum_{i=1}^{n} i \cdot i! = (n+1)! - 1 \). (Recall from calculus that \( i! \) is the product of every positive integer from 1 to \( i \).)

(5) Show that if \( p \) is a prime number then the only solutions of \( x^2 \equiv 1 \mod p \) are \( x \equiv \pm 1 \mod p \) and \( x \equiv (p-1) \mod p \). (You must first show that they are solutions, and then prove that there are no other solutions). Then, find an example where \( n \) is not prime such that \( x^2 \equiv 1 \mod n \) has other solutions besides \( x \equiv \pm 1 \mod n \) and \( x \equiv (n-1) \mod n \).

(6) Prove by mathematical induction that if \( A \) is a set with \( |A| = n \) and \( n \geq 2 \), then \( A \) has \( \frac{n^2-n}{2} \) subsets which contain 2 elements. (Hint: Let \( A = \{1, 2, 3, \ldots, n\} \) and see what the pattern is for small values of \( n \) first.)