(1) (a) Prove that a graph \( G \) is bipartite if and only if every cycle of \( G \) has even length. If \( G \) is bipartite then \( V = U \cup W \), and so every cycle has the form \( x = u_1, w_1, u_2, w_2, \ldots, u_n, w_n, u_n = x \) or \( x = w_1, u_1, w_2, u_2, \ldots, w_n, u_n = x \) where \( u_i \in U \) and \( w_i \in W \). Therefore every cycle has length \( 2n \) which is even. Conversely, suppose every cycle has even length. Choose a vertex \( u \) and put it into \( U \), then since every cycle which passes through \( u \) has even length you can put every vertex which is an even number of steps from \( u \) in \( U \) and every vertex which is an odd number of steps away in \( W \). You can extend this labelling of vertices to every cycle which intersects this cycle because it also has even length. If the graph is not connected you can do them same thing in the other components. Then \( V = U \cup W \) is a partition of the vertex set such that every edge has one vertex in \( U \) and one in \( V \).

(b) Prove that \( Q_n \) is bipartite for every \( n \). The vertices of \( Q_n \) are labelled by the bitstrings of length \( n \), and two vertices are adjacent if they differ in one digit. Therefore every digit in a vertex in a circuit must be changed an even number of times, so the length of every circuit is even.

(2) A graph is planar if you can draw it in the plane (in 2 dimensions) so that no edges cross each other except at vertices.

(a) Show that \( Q_3 \) is planar. Draw one square inside another and join the relevant dots.

(b) Show that \( Q_4 \) is planar. Oops, \( Q_4 \) is not planar.

(3) Show that if \( G \) is a bipartite simple graph with \( v \) vertices and \( e \) edges, then \( e \leq v^2/4 \). If \( V = V_1 \cup V_2 \) then \( v = |V_1| + |V_2| = v_1 + v_2 \) and \( e \leq v_1v_2 \), use calculus/differentiation/completing the square to maximize \( E = v_1v_2 \) with the restriction \( v = v_1 + v_2 \), so \( E = v_1(v - v_1) \) where \( v \) is a constant and \( v_1 \) is a variable.

(4) How many paths are there from 101 to 010 in \( Q_3 \)? Just counting simple paths (otherwise the answer is infinity), all three digits have to end up different, so each digit must change an odd number of times, so each path must have an odd number of edges. The possibilities for the length of the path are 3, 5 and 7. Represent a path as a sequence of \( x, y \) and \( z \) where \( x \) represents an edge which is a change in the first digit, \( y \) in the second, and \( z \) in the third. A path of length 3 has the form \( xyz \) in some order so there are 6 of these. A path of length 5 has the shape/form \( xyxzx \), the \( x \) can be chosen in 3 ways and the \( y \) in 2 ways but then the remaining three positions are determined (otherwise an edge would be used again or the path would reach 010 in three steps.) There are three possible forms for a path of length 7: \( xyxyzyx \) or \( xyxyzxy \) or \( xyxzyxy \) and in each of these can be formed in 6 ways by permuting \( x, y \) and \( z \). So the total is 30. You should probably draw \( Q_3 \) in the plane to convince yourself of these numbers.

(5) Prove that a tree is a bipartite graph. Choose a root \( r \) for the tree and put it in \( V_1 \), then there is a unique simple path from \( r \) to any vertex \( v \). If the path has even length put \( v \in V_1 \) and if it has odd length put \( v \in V_2 \). Then \( V = V_1 \cup V_2 \) is a partition of the vertex set such that every edge has one vertex in \( V_1 \) and one in \( V_2 \).