Directions: You have 90 minutes to answer the following questions. You must show all your work as neatly and clearly as possible and indicate the final answer clearly. You may use a calculator. The last page contains formulas that you might find useful. You may tear that page out. You may choose to have only 8 problems (each worth 12 points) graded or 9 problems (each worth 11) graded or 10 problems (each worth 10) graded. For example if you do not want to have Problem 6 graded, you MUST put an “X” in the Points section of Problem 6.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Possible</th>
<th>Points</th>
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(1) (Page 141, Problem 27) Find all points on the graph of

\[ x^2 + y^2 = 4x + 4y \]

at which the tangent line is horizontal. Show all your work.
(2) (Page 260, Problem 14) When a drug is injected into the bloodstream, its concentration \( C(t) \) at \( t \) minutes after injection is given by a formula of the form

\[
C(t) = M \left( \frac{e^{-bt} - e^{-at}}{a - b} \right)
\]

where \( M, a \) and \( b \) are positive constants, and \( a > b \).

(a) When does the maximum concentration occur? Your answer may contain \( M, \ a \) and/or \( b \).

(b) The function \( C \) has inflections point(s) at \( t = \) ____________.
(3) (MA 1022, Worksheet 6) Find a formula for $y$ as a function of $x$ solving the initial value problem

$$2y'' + py' - p^2 y = 0 \quad y(0) = 0 \quad \text{and} \quad y'(0) = q$$

where $p$ and $q$ are positive constants.
(4) (MA 1112, Worksheet 4) Decide whether each of the following statements is True or False. (You need not show your work.)

(a) If \( f \) is a positive continuous function such that \( \int_{0}^{1} f(x) \, dx \) diverges, then \( \int_{0}^{1} f(\sqrt{x}) \, dx \) also diverges.

(b) If the function \( g \) is positive and continuous on the interval \((0, 1)\) such that \( \int_{1}^{\infty} g(x) \, dx \) converges, then \( \int_{1}^{\infty} \frac{g(x)}{\sqrt{x}} \, dx \) also converges.

(c) \( \int_{-2}^{2} \frac{1}{x^5} \, dx = 0. \)

(d) \( \int_{1}^{\infty} \frac{1}{x^2 + x} \, dx \) converges.
(5) (MA 1112, Worksheet III) Let \( G(x) = \int_0^x f(t) \, dt \), where \( f \) is a continuous function. Some of the values of \( G \) and its derivatives are given in the table below.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( G(x) )</th>
<th>( G'(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-1)</td>
<td>1.2</td>
<td>(-3)</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>(-0.5)</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>( \pi )</td>
<td>1</td>
<td>7</td>
</tr>
<tr>
<td>( 3\pi/4 )</td>
<td>(-2)</td>
<td>0</td>
</tr>
</tbody>
</table>

Evaluate the following definite integrals. Write “NEI” if there is not enough information to give the answer.

(a) \( \int_{3\pi/4}^{\pi} \frac{f(\tan(\theta))}{\cos^2(\theta)} \, d\theta \).

(b) \( \int_{-1}^{1} x f'(x) \, dx \).
(6) (MA 1122, Worksheet 3) Suppose you know that all derivatives of some function $f$ exist at 0, and that the Taylor series for $f$ about $a = 0$ is

$$1 - \frac{x^2}{2} + \frac{x^4}{4} - \frac{x^6}{6} + \frac{x^8}{8} - ...$$

Fill in the blanks.

(a) The function has a __________________________ (local max or local min) at $x = 0$, the value of the local extremum is __________________________.

(b) $f^{(505)}(0) = __________________________$

(c) $f^{(808)}(0) = __________________________$

(d) If we use tangent line approximation to estimate the value of $f(0.03)$, then the approximate value is __________________________.
(7) (Page 382, Problem 10) Let $\mathcal{R}$ be the region bounded by the graphs of $y = -x^2 + 1$ and $y = -2x^2 + 2$. Set up, but do not evaluate, the definite integral which gives the volume of the solid generated by revolving $\mathcal{R}$ about the line $y = -2$. 
(8) (Page 434, Problem 12) Use the second order Taylor polynomial centered at \( x = 0 \) of the function \( f(x) = \sqrt{3x + 9} \) to approximate \( \sqrt{9.027} \).
(9) (Page 297, Problem 13) The figure below shows the graph of the derivative $g'(x)$ of a function $g(x)$. It is given that $g(0) = 50$. Sketch the graph of $g(x)$, showing all critical points and inflection points of $g$ and giving their coordinates.
(10) (Practice Exam) Solve the following differential equation:
\[ y' + 7y = \sin(x). \]
Useful formulas

• **Geometry Formulas**
  Here $V$ is the volume, $S$ is the surface area, $h$ is the height and $r$ is the radius.

  - Cylinder: $V = \pi r^2 h$, $S = 2\pi rh$
  - Cone: $V = \frac{1}{3} \pi r^2 h$
  - Sphere: $V = \frac{4}{3} \pi r^3$, $S = 4\pi r^2$

• **Physics formulas:**
  - The acceleration due to gravity, $g$: $g = 9.8 \text{m/sec}^2$, or $g = 32 \text{ft/sec}^2$.
  - Force = mass $\times$ acceleration
  - Work = Force $\times$ distance
  - The center of mass, $\bar{x}$, of an object lying on the $x$-axis between $x = a$ and $x = b$, with mass density $\delta(x)$ is given by $\bar{x} = \frac{\int_a^b x \delta(x) \, dx}{\text{total mass}}$

  - Arc length of a curve $y = f(x)$ from $x = a$ to $x = b$: $L = \int_a^b \sqrt{1 + (f'(x))^2} \, dx$

• **Integration by Parts:**
  $$\int u \, dv = uv - \int v \, du$$

• **Numerical Approximations:**
  - TRAP$(n) = \frac{\text{LEFT}(n) + \text{RIGHT}(n)}{2}$
  - SIMP$(n) = \frac{2 \text{MID}(n) + \text{TRAP}(n)}{3}$

• Summary of solutions to $y'' + by' + cy = 0$.
  - If $b^2 - 4c > 0$, then $r_1$ and $r_2$ are two distinct solutions of the characteristic equation and
    $y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$,
    where $C_1$ and $C_2$ are constants.
  - If $b^2 - 4c = 0$, then there is only one solution of the characteristic equation, $r = -b/2$, and
    $y = C_1 t e^{rt} + C_2 e^{rt}$.
  - If $b^2 - 4c < 0$, then the solutions of the characteristic equation are of the form $r = \alpha \pm \beta i$ and
    $y = C_1 e^{\alpha t} \cos(\beta t) + C_2 e^{\alpha t} \sin(\beta t)$.

• **nth degree Taylor Polynomial of $f(x)$ centered at $x = a$:**
  $$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n$$

• **Taylor series of $f(x)$ centered at $x = a$:**
  $$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3 + \cdots$$
• Taylor Series of important functions:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \text{ for } -1 < x < 1$$

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \text{ for } -1 < x \leq 1$$

$$(1 + x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots \text{ for } -1 < x < 1$$

• Finite Geometric Series:

$$a + ax + ax^2 + \cdots + ax^{n-1} = \frac{a(1 - x^n)}{1-x}$$

• Infinite Geometric Series:

$$a + ax + ax^2 + \cdots = \frac{a}{1-x} \text{ for } |x| < 1$$

• Ratio Test:
For the series $\sum a_n$, suppose,

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L.$$

– If $L < 1$, then the series converges.
– If $L > 1$, then the series diverges.
– If $L = 1$, then the test fails.

• Differentiation formulas

<table>
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<tr>
<th>( \frac{d}{dx}(x^n) = nx^{n-1} )</th>
<th>( \frac{d}{dx}(e^x) = e^x )</th>
<th>( \frac{d}{dx}(a^x) = (\ln a)a^x )</th>
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</thead>
<tbody>
<tr>
<td>( \frac{d}{dx}(\ln</td>
<td>x</td>
<td>) = \frac{1}{x} )</td>
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<tr>
<td>( \frac{d}{dx}(\tan(x)) = \sec^2 x )</td>
<td>( \frac{d}{dx}(\sec(x)) = \sec x \tan x )</td>
<td>( \frac{d}{dx}(\csc(x)) = -\csc x \cot x )</td>
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<tr>
<td>( \frac{d}{dx}(\arcsin(x)) = \frac{1}{\sqrt{1-x^2}} )</td>
<td>( \frac{d}{dx}(\arccos(x)) = \frac{-1}{\sqrt{1-x^2}} )</td>
<td>( \frac{d}{dx}(\arctan(x)) = \frac{1}{1+x^2} )</td>
</tr>
</tbody>
</table>
Here $a, b, c, d$ are constants.

A Short Table of Indefinite Integrals

I. Basic Functions

1. $\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C$, $(n \neq -1)$

2. $\int \frac{1}{x} \, dx = \ln |x| + C$

3. $\int ax \, dx = \frac{1}{\ln a} ax + C$

4. $\int \ln x \, dx = x \ln x - x + C$

5. $\int \sin ax \, dx = -\frac{1}{a} \cos ax + C$

6. $\int \cos ax \, dx = \frac{1}{a} \sin ax + C$

7. $\int \tan ax \, dx = -\frac{1}{a} \ln |\cos ax| + C$

II. Products of $e^x$, $\cos x$, and $\sin x$

8. $\int e^{ax} \sin(bx) \, dx = \frac{1}{a^2 + b^2} e^{ax} [a \sin(bx) - b \cos(bx)] + C$

9. $\int e^{ax} \cos(bx) \, dx = \frac{1}{a^2 + b^2} e^{ax} [a \cos(bx) + b \sin(bx)] + C$

10. $\int \sin(ax) \sin(bx) \, dx = \frac{1}{b^2 - a^2} [a \cos(ax) \sin(bx) - b \sin(ax) \cos(bx)] + C$, $a \neq b$

11. $\int \cos(ax) \cos(bx) \, dx = \frac{1}{b^2 - a^2} [b \cos(ax) \sin(bx) - a \sin(ax) \cos(bx)] + C$, $a \neq b$

12. $\int \sin(ax) \cos(bx) \, dx = \frac{1}{b^2 - a^2} [b \sin(ax) \sin(bx) + a \cos(ax) \cos(bx)] + C$, $a \neq b$

III. Product of Polynomial $p(x)$ with $\ln x, e^x$, $\cos x$, and $\sin x$

13. $\int x^n \ln x \, dx = \frac{1}{n+1} x^{n+1} \ln x - \frac{1}{(n+1)^2} x^{n+1} + C$, $n \neq -1, x > 0$

14. $\int p(x) e^{ax} \, dx = \frac{1}{a} p(x) e^{ax} - \frac{1}{a^2} p'(x) e^{ax} + \frac{1}{a^3} p''(x) e^{ax} - \cdots + C$

\hspace{2cm} (+ - + - + - + \ldots) (signs alternate)

15. $\int p(x) \sin ax \, dx = -\frac{1}{a} p(x) \cos(ax) + \frac{1}{a^2} p'(x) \sin(ax) + \frac{1}{a^3} p''(x) \cos(ax) - \cdots + C$

\hspace{2cm} (- + + - + - + \ldots) (signs alternate in pairs)

16. $\int p(x) \cos ax \, dx = \frac{1}{a} p(x) \sin(ax) + \frac{1}{a^2} p'(x) \cos(ax) - \frac{1}{a^3} p''(x) \sin(ax) - \cdots + C$

\hspace{2cm} (+ + - - + + - \ldots) (signs alternate in pairs)
IV. Integer Powers of $\sin x$ and $\cos x$

17. $\int \sin^n x \, dx = \frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx, \quad n \text{ positive}$

18. $\int \cos^n x \, dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x \, dx, \quad n \text{ positive}$

19. $\int \frac{1}{\sin^m x} \, dx = -\frac{1}{m-1} \frac{\cos x}{\sin^{m-1} x} + \frac{m-2}{m-1} \int \frac{1}{\sin^{m-2} x} \, dx, \quad m \neq 1, m \text{ positive}$

20. $\int \frac{1}{\sin x} \, dx = \frac{1}{2} \ln \left| \frac{\cos x - 1}{\cos x + 1} \right| + C$

21. $\int \frac{1}{\cos^m x} \, dx = \frac{1}{m-1} \frac{\sin x}{\cos^{m-1} x} + \frac{m-2}{m-1} \int \frac{1}{\cos^{m-2} x} \, dx, \quad m \neq 1, m \text{ positive}$

22. $\int \frac{1}{\cos x} \, dx = \frac{1}{2} \ln \left| \frac{\sin x + 1}{\sin x - 1} \right| + C$

23. $\int \sin^m x \cos^n x \, dx$ :
   If $n$ is odd, let $w = \sin x$.
   If both $m$ and $n$ are even and non-negative, convert all to $\sin x$ or all to $\cos x$ (using $\sin^2 x + \cos^2 x = 1$), and use IV-17 or IV-18.
   If $m$ and $n$ are even and one of them is negative, convert to whichever function is in the denominator and use IV-19 or IV-21.
   The case in which both $m$ and $n$ are even and negative is omitted.

V. Quadratic in the Denominator

24. $\int \frac{1}{x^2 + a^2} \, dx = \frac{1}{a} \arctan \left( \frac{x}{a} \right) + C, \quad a \neq 0$

25. $\int \frac{bx + c}{x^2 + a^2} \, dx = \frac{b}{2} \ln |x^2 + a^2| + \frac{c}{a} \arctan \left( \frac{x}{a} \right) + C, \quad a \neq 0$

26. $\int \frac{1}{(x-a)(x-b)} \, dx = \frac{1}{(a-b)} \ln |x-a| - \ln |x-b| + C, \quad a \neq b$

27. $\int \frac{cx + d}{(x-a)(x-b)} \, dx = \frac{1}{(a-b)} [(ac + d) \ln |x-a| - (bc + d) \ln |x-b|] + C, \quad a \neq b$

VI. Integrands involving $\sqrt{a^2 + x^2}$, $\sqrt{a^2 - x^2}$, $\sqrt{x^2 - a^2}$, $a > 0$

28. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \left( \frac{x}{a} \right) + C$

29. $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln |x + \sqrt{x^2 \pm a^2}| + C$

30. $\int \sqrt{a^2 \pm x^2} \, dx = \frac{1}{2} \left( x \sqrt{a^2 \pm x^2} + a^2 \int \frac{1}{\sqrt{a^2 \pm x^2}} \, dx \right) + C$

31. $\int \sqrt{x^2 - a^2} \, dx = \frac{1}{2} \left( x \sqrt{x^2 - a^2} + a^2 \int \frac{1}{\sqrt{x^2 - a^2}} \, dx \right) + C$